LARGE TIME PROBABILITY OF FAILURE IN DIFFUSIVE SEARCH WITH RESETTING FOR A RANDOM TARGET IN $\mathbb{R}^d$–A FUNCTIONAL ANALYTIC APPROACH

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ABSTRACT. We consider a stochastic search model with resetting for an unknown stationary target $a \in \mathbb{R}^d$, $d \geq 1$, with known distribution μ. The searcher begins at the origin and performs Brownian motion with diffusion coefficient D. The searcher is also equipped with an exponential clock with rate $r > 0$, so that if it has failed to locate the target by the time the clock rings, then its position is reset to the origin and it continues its search anew from there. In dimension one, the target is considered located when the process hits the point a, while in dimensions two and higher, one chooses an $\epsilon_0 > 0$ and the target is considered located when the process hits the ϵ_0-ball centered at a. Denote the position of the searcher at time t by $X(t)$, let τ_a denote the time that a target at a is located, and let $P_0^{d;(r,0)}$ denote probabilities for the process starting from 0. Taking a functional analytic point of view, and using the generator of the Markovian search process and its adjoint, we obtain precise estimates, with control on the dependence on a, for the asymptotic behavior of $P_0^{d;(r,0)}(\tau_a > t)$ for large time, and then use this to obtain large time estimates on $\int_{\mathbb{R}^d} P_0^{d;(r,0)}(\tau_a > t)d\mu(a)$, the probability that the searcher has failed up to time t to locate the random target, for a variety of families of target distributions μ. Specifically, for $B, l > 0$ and $d \in \mathbb{N}$, let $\mu_{B,l}^{(d)} \in \mathcal{P}(\mathbb{R}^d)$ denote any target distribution with density $\mu_{B,l}^{(d)}(a)$ that satisfies

$$\lim_{|a| \to \infty} \frac{\log \mu_{B,l}^{(d)}(a)}{|a|^l} = -B.$$

Then we prove that

$$\lim_{t \to \infty} \frac{1}{(\log t)^l} \log \int_{\mathbb{R}^d} P_0^{d;(r,0)}(\tau_a > t)\mu_{B,l}^{(d)}(da) = -B\left(\frac{D}{2r}\right)^{\frac{l}{2}}.$$

The result is independent of the dimension. In particular, for example, if the target distribution is a centered Gaussian of any dimension with variance σ^2, then for any $\delta > 0$, the probability of not locating the target by time t falls in the interval $\left(e^{-(1+\delta)\frac{D}{4r\sigma^2}(\log t)^2}, e^{-(1-\delta)\frac{D}{4r\sigma^2}(\log t)^2}\right)$, for sufficiently large t.

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1. Introduction and Statement of Results

The use of resetting in search problems is a common phenomenon in various contexts. For example, in everyday life, one might be searching for some target, such as a face in a crowd or a misplaced object. After having searched unsuccessfully for a while, there is a tendency to return to the starting point and begin the search anew. Other contexts where search problems frequently involve resetting include animal foraging [2, 25], proteins searching for target sites on DNA molecules [3, 6, 15] and internet search algorithms.

Over the past decade or so, a variety of stochastic processes with resetting have attracted much attention, mainly in the physics literature. See [12] for a rather comprehensive, recent overview. Prominent among such processes is the diffusive search process with resetting, the process we consider in this paper. Consider a random stationary target $a \in \mathbb{R}^d$ with known distribution μ, and consider a searcher who sets off from the origin, and performs d-dimensional Brownian motion with diffusion coefficient D. The searcher is also equipped with an exponential clock with rate r, so that if it has failed to locate the target by the time the clock rings, then its position is reset to the origin and it continues its search anew from there. In dimension one, the target is considered "located" when the process hits the point a, while in dimensions two and higher, one chooses an $\epsilon_0 > 0$ and the target is considered "located" when the process hits the ϵ_0-ball centered at a. One may be interested in several statistics, the most important ones being the expected time to locate the target and the probability of failing to reach the target after a large time. See, for example, [8, 9, 10, 11, 13, 18, 17, 7, 23] for a sampling of articles on this model and related ones.

The objective of this paper is to give a rigorous analysis of the latter of these two statistics, from a functional analytic point of view, using the generator of the Markovian search process and its adjoint. However, we begin with some comments concerning the first of these statistics. Without the resetting, the expected time to locate the target at any fixed $a \in \mathbb{R}^d - \{0\}$ is infinite [21]. With the resetting, the expected time to locate the target at $a \in \mathbb{R}^d$ is finite. In dimension one it is given by $\frac{e^{\sqrt{\frac{2r}{D}}|a|} - 1}{r}$, $a \in \mathbb{R}$, [9] while in dimensions $d \geq 2$ it is given explicitly in terms of the modified Bessel function of the second kind, $K_{\frac{d-2}{2}}$ [10]. From the above formula in one-dimension, the expected time to locate the random target is

$\int_{-\infty}^{\infty} \frac{e^{\sqrt{\frac{2r}{D}}|a|}-1}{r}\mu(da)$. In particular, in order for this expected time to be finite, the target distribution μ must possess some exponential moments. A similar phenomenon holds in higher dimensions. In [22], a spatially dependent exponential resetting rate was considered in the one-dimensional case, and it was shown that for any distribution μ with finite lth moment, for some $l > 2$, one can choose a spatially dependent resetting rate so that the expected time to locate the random target is finite.

In this paper we consider a constant resetting rate r. Before discussing our results concerning the large time probability that the searcher fails to locate the target, we give a more formal mathematical definition of the model. The process $X(t)$ on $\mathbb{R}^d$ is defined as follows. The process starts from $0 \in \mathbb{R}^d$ and performs d-dimensional Brownian motion with diffusion coefficient D, until a random clock rings. This random clock has an exponential distribution with parameter r, so the probability that it has not rung by time t is e^{-rt}. When the clock rings, the process is instantaneously reset to its initial position 0, and continues its search afresh with an independent resetting clock, and the above scenario is repeated, etc. We define the process so that it is right-continuous. Denote probabilities and expectations for the process starting at $x \in \mathbb{R}^d$ by $P_x^{d;(r,0)}$ and $E_x^{d;(r,0)}$ respectively. The pair $(r,0)$ in the notation refers to the resetting rate r and the resetting position 0. (For the analysis in the multidimensional case, we will need to consider resetting to a point different than 0.) From the above description, it follows that $X(t)$ is a Markov process whose generator $L^{d;(r,0)}$, restricted to appropriate functions u, satisfies

$$(1.1) \qquad L_d^{d;(r,0)}u(x) = \frac{D}{2}\Delta u(x) + r\big(u(0) - u(x)\big).$$

(See the proof of Proposition 3 and Proposition 2-Bessel for more details.)

Fix $\epsilon_0 > 0$ once and for all. Let

$$(1.2) \qquad \tau_a = \begin{cases} \inf\{t \geq 0 : X(t) = a\}, \ d = 1; \\ \inf\{t \geq 0 : |X(t) - a| \leq \epsilon_0\}, \ d \geq 2 \end{cases}$$

denote the time at which a target at $a \in \mathbb{R}^d$ is located. In this paper, we study the asymptotic behavior as $t \to \infty$ of $P_0^{d;(r,0)}(\tau_a > t)$, the probability that the resetting process has not located a target at a by time t, and then use this to analyze the asymptotic behavior as $t \to \infty$ of $\int_{-\infty}^{\infty} P_0^{d;(r,0)}(\tau_a > t)\mu(da)$, the probability that the searcher has failed up to time t to locate

the random target, distributed according to $\mu \in \mathcal{P}(\mathbb{R}^d)$. The asymptotic behavior of $P_0^{d;(r,0)}(\tau_a > t)$ has already been investigated in [8] for the one-dimensional case and in [10] for the multi-dimensional case, using the method of inverse Laplace transforms. The mathematics there is a bit informal. Using our functional analytic approach, the basic asymptotic behavior we obtain is the same as in those papers, however the form in which we obtain it gives us explicit control over the dependence of this behavior on a, in contrast to the state of affairs in the above-mentioned papers, as far as this author can tell. We elaborate on this more in the next paragraph. This control is crucial for the next step, which is the main point of the paper, namely the analysis of $\int_{-\infty}^{\infty} P_0^{d;(r,0)}(\tau_a > t)\mu(da)$. In addition, the form in which we obtain our estimate on $P_0^{d;(r,0)}(\tau_a > t)$ allows for greater understanding of the underlying probabilistic mechanisms at work. Furthermore, we identify explicitly a number of spectral theoretic quantities, such as the principal eigenfunctions of the operator and its adjoint, and this might be of some independent interest. The papers in the physics literature have not studied the asymptotic behavior of $\int_{-\infty}^{\infty} P_0^{d;(r,0)}(\tau_a > t)\mu(da)$, the probability that the searcher has failed to locate the random target by time t; thus, our work on this is entirely new.

An asymptotic formula of the form $P_0^{d;(r,0)}(\tau_a > t) \sim c(a,t)e^{-\lambda_0(r,0;a)t}$ is obtained both in [8, 10] and in this paper, where $\lambda_0(r,0;a)$ satisfies a certain implicit equation, which allows for its asymptotic analysis as $a \to \infty$. In [8, 10], $\lambda_0(r,0;a)$ arises from the inverse Laplace transform method, while in this paper, it arises as a certain principal eigenvalue. However, the term $c(a,t)$ is not analyzed sufficiently for our needs in [8, 10]. In our paper, we obtain the term $c(a,t)$ explicitly in terms of an expectation involving the search process, and this allows us sufficient control over $c(a,t)$ in order to study the asymptotic behavior of $\int_{-\infty}^{\infty} P_0^{d;(r,0)}(\tau_a > t)\mu(da)$ for certain families of target distributions μ.

Before stating the main results, we describe a side result which will follow readily from the results concerning $P_0^{d;(r,0)}(\tau_a > t)$. The Brownian motion without resetting corresponds to setting $r = 0$; let $P_x^{d;(0)}$ and $E_x^{d;(0)}$ denote probabilities and expectations for the Brownian motion without resetting starting from $x \in \mathbb{R}^d$. As already noted, for fixed $a \in \mathbb{R}^d - \{0\}$, the expected time to locate a target at a by a Brownian motion without resetting

is infinite, but for the Brownian motion with resetting it is finite. However, the one-dimensional (two-dimensional) Brownian motion without resetting reaches distant points (the ϵ_0-neighborhood of distant points) much more quickly than does one-dimensional (two-dimensional) Brownian motion with resetting. (Of course, in three dimensions and higher, Brownian motion without resetting has a positive probability of never reaching the ϵ_0-neighborhood of a point.) In the one-dimensional case without resetting, using Brownian scaling (or alternatively, the reflection principle), one can readily show that

$$\lim_{t\to\infty} P_0^{1;(0)}(\tau_{a_t} > t) = \begin{cases} 0, & \text{if } \lim_{t\to\infty} \frac{|a_t|}{\sqrt{t}} = 0; \\ 1, & \text{if } \lim_{t\to\infty} \frac{|a_t|}{\sqrt{t}} = \infty. \end{cases}$$

In the two-dimensional case without resetting, we have the following result.

Proposition 1.

$$\lim_{t\to\infty} P_0^{2;(0)}(\tau_{a_t} > t) = 0, \ \ if \ \lim_{t\to\infty} \frac{|a_t|}{t^\delta} = 0, \ for \ all \ \delta > 0;$$

$$(1.3) \quad \liminf_{t\to\infty} P_0^{2;(0)}(\tau_{a_t} > t) > 0, \ \ if \ \liminf_{t\to\infty} \frac{|a_t|}{t^\delta} > 0, \ for \ some \ \delta \in (0, \tfrac{1}{2}];$$

$$\lim_{t\to\infty} P_0^{2;(0)}(\tau_{a_t} > t) = 1, \ \ if \ \lim_{t\to\infty} \frac{|a_t|}{t^{\frac{1}{2}}} = \infty.$$

On the other hand, we will prove the following result for the Brownian motion with resetting.

Proposition 2. *For $d = 1$,*

$$(1.4) \quad \lim_{t\to\infty} P_0^{1;(r,0)}(\tau_{a_t} > t) = \begin{cases} 0, & \text{if } \lim_{t\to\infty}(|a_t| - \sqrt{\frac{D}{2r}} \log t) = -\infty; \\ 1, & \text{if } \lim_{t\to\infty}(|a_t| - \sqrt{\frac{D}{2r}} \log t) = \infty. \end{cases}$$

For $d \geq 2$,

$$(1.5)$$

$$\lim_{t\to\infty} P_0^{d;(r,0)}(\tau_{a_t} > t) = 0, \ if \ \lim_{t\to\infty}\left(|a_t| - \sqrt{\frac{D}{2r}} \log t + \gamma \log\log t\right) = -\infty,$$

for some $\gamma > \dfrac{d-1}{2}\sqrt{\dfrac{D}{2r}}$;

$$\lim_{t\to\infty} P_0^{d;(r,0)}(\tau_{a_t} > t) = 1, \ if \ \lim_{t\to\infty}\left(|a_t| - \sqrt{\frac{D}{2r}} \log t + \frac{d-1}{2}\sqrt{\frac{D}{2r}} \log\log t\right) = \infty.$$

We now turn to the main results. We will be interested in the behavior of the process with D and r fixed. In our notation, we suppress all dependence

on D (except in Corollary 1 and Corollary 1-Bessel, where the dependence of certain constants on D is indicated), but indicate the dependence on r. We begin by stating our central result, which concerns $\int_{-\infty}^{\infty} P_0^{d;(r,0)}(\tau_a > t)\mu(da)$, the probability that the searcher has failed to locate the random target by time t.

For $B, l > 0$ and $d \in \mathbb{N}$, let $\mu_{B,l}^{(d)} \in \mathcal{P}(\mathbb{R}^d)$ denote any target distribution with density $\mu_{B,l}^{(d)}(a)$ that satisfies

$$(1.6) \qquad \lim_{|a|\to\infty} \frac{\log \mu_{B,l}^{(d)}(a)}{|a|^l} = -B.$$

Theorem 1. *Let $\mu_{B,l}^{(d)} \in \mathcal{P}(\mathbb{R}^d)$ be a distribution with density satisfying (1.6). Then $\int_{\mathbb{R}^d} P_0^{d;(r,0)}(\tau_a > t)\mu_{B,l}^{(d)}(da)$, the probability that the searcher with resetting fails to locate the random target with distribution $\mu_{B,l}^{(d)}$ by time t, satisfies*

$$(1.7) \qquad \lim_{t\to\infty} \frac{1}{(\log t)^l} \log \int_{\mathbb{R}^d} P_0^{d;(r,0)}(\tau_a > t)\mu_{B,l}^{(d)}(da) = -B\left(\frac{D}{2r}\right)^{\frac{l}{2}}.$$

Remark. Unlike all of the other results in this paper, the result in Theorem 1 is independent of the dimension.

Example 1. Consider a target distribution of the form (1.6) with $l = 1$. In particular, if $d = 1$, this situation includes the two-sided, symmetric exponential distributions, whose densities are of the form $Be^{-B|x|}$, $B > 0$. One has that for any $\delta > 0$, the probability of not locating the target by time t falls in the interval $(t^{-(1+\delta)B\sqrt{\frac{D}{2r}}}, t^{-(1-\delta)B\sqrt{\frac{D}{2r}}})$, for sufficiently large t.

Example 2. Consider a centered Gaussian target distribution in any dimension, with variance σ^2. This distribution is of the form (1.6) with $l = 2$ and $B = \frac{1}{2\sigma^2}$. For such a target distribution, for any $\delta > 0$, the probability of not locating the target by time t falls in the interval $\left(e^{-(1+\delta)\frac{D}{4r\sigma^2}(\log t)^2}, e^{-(1-\delta)\frac{D}{4r\sigma^2}(\log t)^2}\right)$, for sufficiently large t.

For the rest of the results, we need to treat separately the one-dimensional and the multi-dimensional cases. We begin with the one-dimensional case. We present a series of results which culminates in a formula for $P_0^{1;(r,0)}(\tau_a >$

t) of the form $c(a,t)e^{-\lambda_0(r,0;a)t}$, where $\lambda_0(r,0;a)$ is a certain principal eigen-value and $c(a,t)$ is given in terms of a certain conditional expectation (Theorem 3), and a result which estimates $c(a,t)$ for large a, uniformly in t (Proposition 5).

For $a \neq 0$, let $T_t^{(r,0;a)}$ denote the semigroup corresponding to the Markov process $X(t)$ that is killed upon reaching a. If $a > 0$, then

$$(1.8) \qquad T_t^{(r,0;a)}f(x) = E_x^{1;(r,0)}(f(X(t)); \tau_a > t), \ x \in (-\infty, a], \ t \geq 0,$$

for bounded functions f defined on $(-\infty, a)$. For $a < 0$, we have the corresponding formula with $x \in [a, \infty)$. From now on we will assume that $a > 0$; of course all the results also hold for $a < 0$, mutatis mutandis. Let $[-\infty, a]$ denote the one-point compactification of $(-\infty, a]$, obtained by adding the point at $-\infty$, and let $C_{0_a}([-\infty, a])$ denote the space of continuous functions on $[-\infty, a]$ which vanish at a. (Note that this space is equivalent to the space of continuous functions u on $(-\infty, a)$ which satisfy $\lim_{x \to -\infty} u(x)$ exists and $\lim_{x \to a} u(x) = 0$.) We will prove the following proposition. As usual, $C_b^2((-\infty, a))$ denotes the space of functions defined on $(-\infty, a)$ which have two continuous and bounded derivatives.

Proposition 3. *For $a, r > 0$ and all $t > 0$, the semigroup operator $T_t^{(r,0;a)}$ is compact from $C_{0_a}([-\infty, a])$ to $C_{0_a}([-\infty, a])$. Furthermore, its generator, which we denote by $L^{(r,0;a)}$, is an extension of the operator $L^{1;(r,0)}$ in (1.1) defined on $C_b^2((-\infty, a)) \cap \{f : f, L^{1;(r,0)}f \in C_{0_a}([-\infty, a])\}$.*

From Proposition 3 it follows that the generator $L^{(r,0;a)}$ has a compact resolvent and consequently a principal eigenvalue, which we denote by $\lambda_0(r,0;a)$. The following theorem and corollary concern this principal eigenvalue and the corresponding principal eigenfunction.

Theorem 2. *Let $a > 0$. The principal eigenvalue $\lambda_0(r,0;a)$ of the generator $L^{(r,0;a)}$ of the semigroup $T_t^{(r,0;a)}$ is the unique solution $\lambda \in (0,r)$ of the equation*

$$(1.9) \qquad\qquad \lambda = r \exp\left(-a\sqrt{\frac{2}{D}(r-\lambda)}\right).$$

A corresponding principal eigenfunction $u_{r,0;a}$ is given by

(1.10)

$$u_{r,0;a}(x) = \frac{r}{r - \lambda_0(r,0;a)}\left(1 - \exp\left(-\sqrt{\frac{2\left(r - \lambda_0(r,0;a)\right)}{D}}\,(a-x)\right)\right), \ x \le a.$$

Corollary 1.

(1.11) $$re^{-\sqrt{\frac{2r}{D}}a} \le \lambda_0(r,0;a) \le re^{-c\sqrt{\frac{2r}{D}}a},$$

where $c = c(r,a,D) \in (0,1)$ and $\lim_{a\to\infty} c(r,a,D) = 1$.

For the statement and proof of Theorem 3 below we need to introduce the adjoint semigroup $\tilde{T}_t^{(r,0;a)}$ to the semigroup $T_t^{(r,0;a)}$. Since $T_t^{(r,0;a)}$ is defined on the Banach space $C_{0_a}([-\infty,a])$, the adjoint $\tilde{T}_t^{(r,0;a)}$ operates on the dual space of bounded linear functions on $C_{0_a}([-\infty,a])$. Since $[-\infty,a]$ with the one-point compactification topology is compact, this dual space is the space of finite signed measures on $[-\infty,a)$ [24, p.28]. Recall that a finite signed measure ν is of the form $\nu = \nu^+ - \nu^-$, where ν^+, ν^- are finite measures. (The reason the measures are on $[-\infty,a)$ instead of on $[-\infty,a]$ is that $f(a) = 0$, for $f \in C_{0_a}([-\infty,a])$.) Let ν be such a finite signed measure. We can write $\nu = c_+\nu^+ - c_-\nu^-$, where ν^+ and ν^- are probability measures on $[-\infty,a)$ and $c_+, c_- \ge 0$. From (1.8), it follows that

(1.12)
$$\tilde{T}_t^{(r,0;a)}\nu(dy) = c_+ \int_{-\infty}^a \nu^+(dx) P_x^{(r,0;a)}(X(t) \in dy; \tau_a > t) -$$
$$c_- \int_{-\infty}^a \nu^-(dx) P_x^{(r,0;a)}(X(t) \in dy; \tau_a > t).$$

Denote the generator of the adjoint semigroup by $\tilde{L}^{(r,0;a)}$. Of course, this operator has the same principal eigenvalue as does $L^{(r,0;a)}$.

Proposition 4. *The generator $\tilde{L}^{(r,0;a)}$ of $\tilde{T}_t^{(r,0;a)}$ satisfies*

(1.13) $$\tilde{L}^{(r,0;a)}v(y) = \frac{D}{2}v''(y) - rv(y) + r\left(\int_{-\infty}^a v(x)dx\right)\delta_0(y),$$

for v satisfying $v \in C_{0_a}([-\infty,a]) \cap C_b^2((-\infty,a))$ and $\int_{-\infty}^a |v(y)|dy < \infty$. Furthermore, a principal eigenfunction $v_{r,0;a}$ corresponding to the principal

eigenvalue $\lambda_0(r, 0; a)$ is given by

(1.14)

$$
v_{r,0;a}(y) = \begin{cases} \exp\left(\sqrt{\frac{2(r-\lambda_0(r,0;a))}{D}}\, y\right), & y < 0; \\ \dfrac{\exp\left(-\sqrt{\frac{2(r-\lambda_0(r,0;a))}{D}}\,(y-a)\right) - \exp\left(\sqrt{\frac{2(r-\lambda_0(r,0;a))}{D}}\,(y-a)\right)}{\exp\left(\sqrt{\frac{2(r-\lambda_0(r,0;a))}{D}}\, a\right) - \exp\left(-\sqrt{\frac{2(r-\lambda_0(r,0;a))}{D}}\, a\right)}, & 0 \le y \le a. \end{cases}
$$

Remark. The right hand side of (1.13) should be understood as the signed measure whose absolutely continuous part has density $\frac{D}{2}v''(y) - rv(y)$, and whose singular part is $r\left(\int_{-\infty}^{a} v(y)dy\right)\delta_0$.

Here is our result concerning the asymptotic behavior of $P_0^{1;(r,0)}(\tau_a > t)$.

Theorem 3. *Let $a > 0$. Then*

(1.15) $$P_0^{1;(r,0)}(\tau_a > t) = \frac{1}{E_0^{1;(r,0)}(u_{r,0;a}(X(t))|\tau_a > t)} e^{-\lambda_0(r,0;a)\,t},$$

where $u_{r,0;a}$ is as in (1.10), Furthermore,

(1.16)
$$
\lim_{t\to\infty} E_0^{1;(r,0)}(u_{r,0;a}(X(t))|\tau_a > t) = \frac{\int_{-\infty}^{a} u_{r,0;a}(x)v_{r,0;a}(x)dx}{\int_{-\infty}^{a} v_{r,0;a}(x)dx} =
$$
$$
\frac{2e^{qa} - 2 - qa}{2e^{qa}(1 - \frac{\lambda(r,0;a)}{r})^2}, \quad \text{with } q = \sqrt{\frac{2(r - \lambda(r,0;a))}{D}},
$$

where $v_{r,0;a}$ is as in (1.14). Thus, for fixed a,

(1.17)
$$
P_0^{1;(r,0)}(\tau_a > t) \sim \frac{2e^{qa}(1 - \frac{\lambda(r,0;a)}{r})^2}{2e^{qa} - 2 - qa} e^{-\lambda_0(r,0;a)\,t}, \text{ as } t \to \infty,
$$
$$
\text{with } q = \sqrt{\frac{2(r - \lambda(r,0;a))}{D}}.
$$

The following proposition concerns the coefficient multiplying the exponential term in (1.15). It will be needed for the proof of Theorem 1 as well as for the proof of Proposition 2.

Proposition 5.

(1.18) $$\lim_{a\to\infty} E_0^{1;(r,0)}(u_{r,0;a}(X(t))|\tau_a > t) = 1, \text{ uniformly over } t \in (0, \infty),$$

where $u_{r,0;a}$ is as in (1.10).

We now turn to the multi-dimensional case. Recall the definition of τ_a from (1.2). We make a construction to reduce the study of $P_0^{d;(r,0)}(\tau_a > t)$ to a one-dimensional problem. Instead of having the target at $a \in \mathbb{R}^d$ and

having the resetting bring the process to $0 \in \mathbb{R}^d$, we consider the target to be at 0 and have the resetting bring the process to a. If we denote this new process by $\hat{X}(t)$ and denote probabilities by $\hat{P}_x^{d;(r;a)}$, then clearly

$$(1.19) \qquad P_0^{d;(r,0)}(\tau_a > t) = \hat{P}_a^{d;(r,a)}(\hat{\tau}_0 > t),$$

where, consistent with the notation in (1.2),

$$\hat{\tau}_0 = \inf\{t \geq 0 : |\hat{X}(t)| \leq \epsilon_0\}.$$

Now let $Y(t) = |\hat{X}(t)|$. Then $Y(t)$ is the radial part of a d-dimensional Brownian motion with diffusion coefficient D, and it is reset at rate r to $|a|$. That is, $Y(t)$ is a Bessel process with resetting, of order d with diffusion coefficient D. Let

$$\tau_{\epsilon_0}^{(Y)} = \inf\{t \geq 0 : Y(t) = \epsilon_0\}.$$

Denote probabilities and expectations for $Y(t)$ starting at $x > \epsilon_0$ and with resetting to $A \in (\epsilon_0, \infty)$ at rate r by $\mathcal{P}_x^{(r,A)}$ and $\mathcal{E}_x^{(r,A)}$. Then clearly,

$$(1.20) \qquad \hat{P}_a^{d;(r,a)}(\hat{\tau}_0 > t) = \mathcal{P}_{|a|}^{(r,|a|)}(\tau_{\epsilon_0}^{(Y)} > t), \ |a| > \epsilon_0.$$

From (1.19) and (1.20), it follows that for the analysis of $P_0^{d;(r,0)}(\tau_a > t)$, it suffices to study $\mathcal{P}_{|a|}^{(r,|a|)}(\tau_{\epsilon_0}^{(Y)} > t)$.

We now present the analogs of Proposition 3, Theorem 2, Corollary 1, Proposition 4, Theorem 3 and Proposition 5 in the context of the above Bessel process with resetting. We use the same labelling and numbering of theorems, propositions and the corollary as was used in the one-dimensional case, but suffix each of these with "Bessel".

The generator of the Bessel process of order d with diffusion coefficient D is $\frac{D}{2}\frac{d^2}{dx^2} + D\frac{d-1}{2x}\frac{d}{dx}$. Define the operator $\mathcal{L}^{(r,A)}$ by

$$(1.21) \qquad \mathcal{L}^{(r,A)}u(x) = \frac{D}{2}u''(x) + D\frac{d-1}{2x}u'(x) + r\big(u(A) - u(x)\big).$$

For $A > 0$, let $\mathcal{T}_t^{(r,A;\epsilon_0)}$ denote the semigroup corresponding to the Markov process $Y(\cdot)$ with resetting to A at rate r, and which is killed upon reaching ϵ_0. Then

$$(1.22) \qquad \mathcal{T}_t^{(r,A;\epsilon_0)}f(x) = \mathcal{E}_x^{(r,A)}(f(Y(t)); \tau_{\epsilon_0}^{(Y)} > t), \ x \in [\epsilon_0, \infty), \ t \geq 0.$$

Let $[\epsilon_0, \infty]$ denote the one-point compactification of $[\epsilon_0, \infty)$, obtained by adding the point at ∞, and let $C_{0_{\epsilon_0}}([\epsilon_0, \infty])$ denote the space of continuous functions on $[\epsilon_0, \infty]$ which vanish at ϵ_0.

Proposition 2-Bessel. *For $A, r > 0$ and all $t > 0$, the semigroup operator $\mathcal{T}_t^{(r,A;\epsilon_0)}$ is compact from $C_{0_{\epsilon_0}}([\epsilon_0, \infty])$ to $C_{0_{\epsilon_0}}([\epsilon_0, \infty])$. Furthermore, its generator, which we denote by $\mathcal{L}^{(r,A;\epsilon_0)}$, is an extension of the operator $\mathcal{L}^{(r;A)}$ in (1.21) defined on $C_b^2((\epsilon_0, \infty)) \cap \{f : f, f', f'' \in C_{0_{\epsilon_0}}([\epsilon_0, \infty])\}$.*

From Proposition 2-Bessel, it follows that the generator $\mathcal{L}^{(r,A;\epsilon_0)}$ has a compact resolvent and consequently a principal eigenvalue, which we denote by $\lambda_0(r, A; \epsilon_0)$. The following theorem and corollary concern this principal eigenvalue and the corresponding principal eigenfunction. In the sequel, K_ν denotes the modified Bessel function of the second kind of order ν. This function decays exponentially at ∞ [1, 26].

Theorem 2-Bessel. *The principal eigenvalue $\lambda_0(r, A; \epsilon_0)$ of the generator $\mathcal{L}^{(r,A;\epsilon_0)}$ of the semigroup $\mathcal{T}_t^{(r,A;\epsilon_0)}$ is the unique solution $\lambda \in (0, r)$ of the equation*

$$(1.23) \qquad \lambda = r\left(\frac{A}{\epsilon_0}\right)^{\frac{2-d}{2}} \frac{K_{\frac{d-2}{2}}\left(\sqrt{(r-\lambda)\frac{2}{D}}\, A\right)}{K_{\frac{d-2}{2}}\left(\sqrt{(r-\lambda)\frac{2}{D}}\, \epsilon_0\right)}.$$

A corresponding principal eigenfunction $\mathcal{U}_{r,A;\epsilon_0}$ is given by

(1.24)

$$\mathcal{U}_{r,A;\epsilon_0}(x) = \frac{r}{r - \lambda_0(r, A; \epsilon_0)}\left(1 - \left(\frac{x}{\epsilon_0}\right)^{\frac{2-d}{2}} \frac{K_{\frac{d-2}{2}}\left(\sqrt{(r-\lambda_0(r, A; \epsilon_0))\frac{2}{D}}\, x\right)}{K_{\frac{d-2}{2}}\left(\sqrt{(r-\lambda_0(r, A; \epsilon_0))\frac{2}{D}}\, \epsilon_0\right)}\right), \ x \geq \epsilon_0.$$

Corollary 1-Bessel. *Let*

$$C(r, \epsilon_0, D) = r^{\frac{3}{4}} \epsilon_0^{\frac{d-2}{2}} \left(\frac{\pi^2 D}{4}\right)^{\frac{1}{4}} \left(K_{\frac{d-2}{2}}\left(\sqrt{\frac{2r}{D}}\, \epsilon_0\right)\right)^{-1}.$$

There exist $C_i(r, A, \epsilon_0, D)$, $i = 1, 2, 3$, satisfying

$$\lim_{A \to \infty} C_i(r, A, \epsilon_0, D) = 1, \ i = 1, 2, 3,$$

such that

$$(1.25) \qquad \begin{aligned} C(r, \epsilon_0, D) C_1(r, A, \epsilon_0, D) A^{\frac{1-d}{2}} e^{-\sqrt{\frac{2r}{D}} A} &\leq \lambda_0(r, A; \epsilon_0) \leq \\ C(r, \epsilon_0, D) C_2(r, A, \epsilon_0, D) A^{\frac{1-d}{2}} e^{-C_3(r, A, \epsilon_0, D)\sqrt{\frac{2r}{D}} A}&. \end{aligned}$$

We now consider the adjoint semigroup $\tilde{\mathcal{T}}_t^{(r,A;\epsilon_0)}$ to the semigroup $\mathcal{T}_t^{(r,A;\epsilon_0)}$. Since $\mathcal{T}_t^{(r,A;\epsilon_0)}$ is defined on the Banach space $C_{0_{\epsilon_0}}([\epsilon_0,\infty])$, the adjoint $\tilde{\mathcal{T}}_t^{(r,A;\epsilon_0)}$ operates on the dual space of bounded linear functions on $C_{0_{\epsilon_0}}([\epsilon_0,\infty])$. Since $[\epsilon_0,\infty]$ with the one-point compactification topology is compact, this dual space is the space of finite signed measures on $(\epsilon_0,\infty]$ [24, p.28]. (The reason the measures are on $(\epsilon_0,\infty]$ instead of on $[\epsilon_0,\infty]$ is that $f(\epsilon_0)=0$, for $f \in C_{0_{\epsilon_0}}([\epsilon_0,\infty])$.) Let ν be such a finite signed measure. We can write $\nu = c_+\nu^+ - c_-\nu^-$, where ν^+ and ν^- are probability measures on $(\epsilon_0,\infty]$ and $c_+, c_- \geq 0$. From (1.22), it follows that

$$(1.26) \quad \begin{aligned} \tilde{\mathcal{T}}_t^{(r,A;\epsilon_0)}\nu(dy) = c_+ \int_{-\infty}^a \nu^+(dx)\mathcal{P}_x^{(r,A)}(Y(t)\in dy; \tau_{\epsilon_0}^{(Y)}>t)- \\ c_- \int_{-\infty}^a \nu^-(dx)\mathcal{P}_x^{(r,A)}(Y(t)\in dy; \tau_{\epsilon_0}^{(Y)}>t). \end{aligned}$$

Denote the generator of the adjoint semigroup by $\tilde{\mathcal{L}}^{(r,A;\epsilon_0)}$. Of course, this operator has the same principal eigenvalue as does $\mathcal{L}^{(r,A;\epsilon_0)}$.

Proposition 3-Bessel. *The generator $\tilde{\mathcal{L}}^{(r,A;\epsilon_0)}$ of $\tilde{\mathcal{T}}_t^{(r,A;\epsilon_0)}$ satisfies*

(1.27)
$$\tilde{\mathcal{L}}^{(r,A;\epsilon_0)}v(y) = \frac{D}{2}v''(y) - D\frac{d-1}{2x}v' + D\frac{d-1}{2x^2}v - rv(y) + r\Big(\int_{\epsilon_0}^\infty v(x)dx\Big)\delta_A(y),$$

for v satisfying $v \in C_{0_{\epsilon_0}}([\epsilon_0,\infty]) \cap C_b^2((\epsilon_0,\infty))$ and $\int_{\epsilon_0}^\infty |v(x)|dx < \infty$. Furthermore, a principal eigenfunction $\mathcal{V}_{r,A;\epsilon_0}$ corresponding to the principal eigenvalue $\lambda_0(r,A;\epsilon_0)$ is given by

(1.28)
$$\mathcal{V}_{r,A;\epsilon_0}(y) = \begin{cases} x^{\frac{d}{2}}K_{\frac{d-2}{2}}(qA)\frac{I_{\frac{d-2}{2}}(q\epsilon_0)K_{\frac{d-2}{2}}(qx)-I_{\frac{d-2}{2}}(qx)K_{\frac{d-2}{2}}(q\epsilon_0)}{I_{\frac{d-2}{2}}(q\epsilon_0)K_{\frac{d-2}{2}}(qA)-I_{\frac{d-2}{2}}(qA)K_{\frac{d-2}{2}}(q\epsilon_0)}, & \epsilon_0 \leq x \leq A; \\ x^{\frac{d}{2}}K_{\frac{d-2}{2}}(qx), & x \geq A, \end{cases}$$

where $q = \sqrt{\frac{2(r-\lambda_0(r,A;\epsilon_0))}{D}}$.

Theorem 3-Bessel. *Let $A > \epsilon_0$. Then*

$$(1.29) \qquad \mathcal{P}_A^{(r,A)}(\tau_{\epsilon_0}^{(Y)}>t) = \frac{1}{\mathcal{E}_A^{(r,A)}(\mathcal{U}_{r,A;\epsilon_0}(Y(t))|\tau_{\epsilon_0}^{(Y)}>t)}e^{-\lambda_0(r,A;\epsilon_0)\,t},$$

where $\mathcal{U}_{r,A;\epsilon_0}$ is as in (1.24), Furthermore,

$$(1.30) \quad \lim_{t\to\infty}\mathcal{E}_A^{(r;A)}(\mathcal{U}_{r,A;\epsilon_0}(Y(t))|\tau_{\epsilon_0}^{(Y)}>t) = \frac{\int_{\epsilon_0}^\infty \mathcal{U}_{r,A;\epsilon_0}(x)\mathcal{V}_{r,A;\epsilon_0}(x)dx}{\int_{\epsilon_0}^\infty \mathcal{V}_{r,A;\epsilon_0}(x)dx},$$

where $\mathcal{V}_{r,A;\epsilon_0}$ is as in (1.28). Thus, for fixed A,

(1.31)
$$\mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > t) \sim \frac{\int_{\epsilon_0}^{\infty} \mathcal{V}_{r,A;\epsilon_0}(x)dx}{\int_{\epsilon_0}^{\infty} \mathcal{U}_{r,A;\epsilon_0}(x)\mathcal{V}_{r,A;\epsilon_0}(x)dx} e^{-\lambda_0(r,A;\epsilon_0)\,t}, \;\; as \; t \to \infty.$$

Proposition 4-Bessel.

(1.32) $\lim_{A\to\infty} \mathcal{E}_A^{(r;A)}(\mathcal{U}_{r,A;\epsilon_0}(Y(t))|\tau_{\epsilon_0}^{(Y)} > t) = 1,$ *uniformly over* $t \in (0,\infty),$

where $\mathcal{U}_{r,A;\epsilon_0}$ is as in (1.24).

In the sections that follow, we prove the results stated above in the order that they appeared, except for Theorem 1 and Propositions 1 and 2, whose proofs appear in that order in the final three sections.

2. Proof of Proposition 3

We begin by showing that $T_t^{(r,0;a)}$ maps $C_{0_a}([-\infty,a])$ to $C_{0_a}([-\infty,a])$. Recall that $P_x^{1;(0)}$ and $E_x^{1;(0)}$ denote probabilities and expectations for the Brownian motion with diffusion parameter D without resetting and started from x. From the definition of the Brownian motion with resetting, we have for $f \in C_{0_a}([-\infty,a])$ and $x \in (-\infty,a)$,

(2.1)
$$T_t^{(r,0;a)}f(x) = e^{-rt}E_x^{1;(0)}(f(X(t);\tau_a > t)+$$
$$\int_0^t ds\, re^{-rs}P_x^{1;(0)}(\tau_a > s)T_{t-s}^{(r,0;a)}f(0).$$

From this it is easy to see that $T_t^{(r,0;a)}$ maps $C_{0_a}([-\infty,a])$ to $C_{0_a}([-\infty,a])$. Indeed, it follows readily from standard results that $\lim_{x\to a} P_x^{1;(0)}(\tau_a > u) = 0$, for all $u > 0$. From this and (2.1) it follows that $\lim_{x\to a} T_t^{(r,0;a)}f(x) = 0$. It also follows readily that for any $N > 0$, $= \lim_{x\to-\infty} P_x^{1;(0)}(X(t) \leq -N, \tau_a > t) = 1$ and that $\lim_{x\to-\infty} P_x^{1;(0)}(\tau_a > s) = 1$, for all $s > 0$. Using these last two facts, if follows from (2.1) that $\lim_{x\to-\infty} T_t^{(r,0;a)}f(x)$ exists for $f \in C_{0_a}([-\infty,a])$. Finally, from (2.1) it follows that $T_t^{(r,0;a)}f(x)$ inherits its continuity for $x \in (-\infty,a)$ from the well-known continuity of $E_x^{1;(0)}(f(X(t);\tau_a > t)$ and $P_x^{1;(0)}(\tau_a > s)$. This completes the proof that $T_t^{(r,0;a)}$ maps $C_{0_a}([-\infty,a])$ to $C_{0_a}([-\infty,a])$.

We now show that $T_t^{(r,0;a)}$ is a compact operator. We write

(2.2) $$E_x^{1;(0)}(f(X(t);\tau_a > t) = \int_{-\infty}^a p^{(a)}(t,x,y)f(y)dy,$$

where $p^{(a)}(t, x, y)$ is the transition sub-probability density for the Brownian motion with diffusion parameter D without resetting, and killed upon hitting a. Using the reflection principle, one can show that

$$
\begin{aligned}
(2.3) \quad & p^{(a)}(t, x, y) = \frac{1}{\sqrt{2\pi Dt}} \exp(-\frac{(y-x)^2}{2Dt}) - \\
& \int_0^t ds \frac{a-x}{\sqrt{2\pi D}s^{\frac{3}{2}}} \exp(-\frac{(a-x)^2}{2Ds}) \frac{1}{\sqrt{2\pi D(t-s)}} \exp(-\frac{(y-a)^2}{2D(t-s)}).
\end{aligned}
$$

Using (2.3) along with (2.2) and (2.1) shows that $T_t^{(r,0;a)}$ maps bounded sets in $C_{0_a}([-\infty, a])$ to equicontinuous and bounded sets in $C_{0_a}([-\infty, a])$. This proves the compactness.

We now turn to the generator. Let $f \in C_b^2((-\infty, a)) \cap \{f : f, L^{1;(r,0)}f \in C_{0_a}([-\infty, a])\}$. Note that from this assumption, it also follows that $f'' \in C([-\infty, a])$. From (2.1) and (2.2), we have

$$
\frac{1}{t}\big(T_t^{(r,0;a)} f(x) - f(x)\big) = \frac{1}{t} e^{-rt} \int_{-\infty}^a p^{(a)}(t, x, y)\big(f(y) - f(x)\big) dy +
$$

$$
\frac{1}{t} \int_0^t ds\, re^{-rs} P_x^{1;(0)}(\tau_a > s)\big(T_{t-s}^{(r,0;a)} f(0) - f(x)\big).
$$

Clearly,

$$
(2.4) \quad \lim_{t\to 0} \frac{1}{t} \int_0^t ds\, re^{-rs} P_x^{(0)}(\tau_a > s)\big(T_{t-s}^{(r,0;a)} f(0) - f(x)\big) = r(f(0) - f(x)).
$$

Also, from (2.3), we have

$$
\begin{aligned}
(2.5) \quad & \lim_{t\to 0} \frac{1}{t} e^{-rt} \int_{-\infty}^a p^{(a)}(t, x, y)\big(f(y) - f(x)\big) dy = \\
& \lim_{t\to 0} \frac{1}{t} e^{-rt} \int_{-\infty}^a \frac{1}{\sqrt{2\pi Dt}} \exp(-\frac{(y-x)^2}{2Dt})\big(f(y) - f(x)\big) dy = \frac{D}{2} f''(x).
\end{aligned}
$$

The first equality in (2.5) follows from the fact that $\int_0^t ds \frac{a-x}{\sqrt{2\pi D}s^{\frac{3}{2}}} \exp(-\frac{(a-x)^2}{2Ds}) = o(t)$ as $t \to 0$. When the term e^{-rt} is absent, the second equality in (2.5) is the classical calculation for the generator of Brownian motion, obtained by writing f in a Taylor series with remainder in the form

$$
(2.6) \quad f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2}(y-x)^2 + \frac{f''(c_y) - f''(x)}{2}(y-x)^2.
$$

It is easy to show that the equality still holds with e^{-rt} present since this term approaches 1 when $t \to 0$. From (2.4) and (2.5) we obtain

$$
(2.7) \quad \lim_{t\to 0} \frac{1}{t}\big(T_t^{(r,0;a)} f(x) - f(x)\big) = \frac{D}{2} f''(x) + r(f(0) - f(x)) = (L^{1;(r,0)} f)(x).
$$

By assumption, $L^{1;(r,0)}f \in C_{0_a}([-\infty, a])$. Furthermore, since $f \in C([-\infty, a])$, it is uniformly continuous on $[-\infty, a]$, and consequently it follows that the convergence in (2.4) is uniform. Also, since $f'' \in C([-\infty, a])$, it is also uniformly continuous, and thus it follows from (2.6) that the convergence with regard to the second equal sign in (2.5) is uniform. Finally, the fact that $f(a) = 0$ guarantees the uniform convergence to 0 of the difference between the two expressions on either side of the first equal sign in (2.5). Thus, the convergence in (2.7) is uniform. This completes the proof of the calculation of the generator $L^{(r,0;a)}$. $\qquad\square$

3. Proofs of Theorem 2 and Corollary 1

Proof of Theorem 2. As noted after Proposition 3, $L^{(r,0;a)}$ has a compact resolvent. Thus, by Proposition 3 and the Krein-Rutman theorem, it follows that if we find a $\lambda \in \mathbb{R}$ and a function u satisfying

$$\frac{D}{2}u''(x) + r(u(0) - u(x)) = -\lambda u \ \text{ in } (-\infty, a);$$

(3.1) $$\lim_{x \to -\infty} u(x) \text{ exists and is finite;}$$

$$u(a) = 0;$$

$$u > 0 \text{ in } (-\infty, a),$$

then λ is necessarily the principal eigenvalue $\lambda_0(r, 0; a)$, and u is a corresponding principal eigenfunction. In order to solve the above nonstandard, homogenous linear equation involving evaluation at a point, for an appropriate λ, we consider the following standard, inhomogeneous linear equation involving a free parameter $c \in \mathbb{R}$:

$$\frac{D}{2}B''_{c,\lambda} + (\lambda - r)B_{c,\lambda} = -rc, \ x \in (-\infty, a);$$

(3.2) $$\lim_{x \to -\infty} B_{c,\lambda}(x) \text{ exists and is finite;}$$

$$B_{c,\lambda}(a) = 0;$$

$$B_{c,\lambda} > 0 \text{ in } (-\infty, a).$$

We will solve explicitly for $B_{c,\lambda}$, for any c and λ, and then we look for a solution (c, λ) to the equation $B_{c,\lambda}(0) = c$. Note that if (c, λ) solves this equation, then $B_{c,\lambda}$ solves (3.1). It turns out that the set of solutions is of the form $\{(c, \lambda^*) : c \in \mathbb{R}\}$, for a unique λ^*.

Define

(3.3)
$$\overline{B}_{c,\lambda} = B_{c,\lambda} - \frac{rc}{r - \lambda}.$$

Then $B_{c,\lambda}$ solves (3.2) if and only if $\overline{B}_{c,\lambda}$ solves

(3.4)
$$\frac{D}{2}\overline{B}''_{c,\lambda} + (\lambda - r)\overline{B}_{c,\lambda} = 0, \ x \in (-\infty, a);$$

$$\lim_{x \to -\infty} \overline{B}_{c,\lambda}(x) \text{ exists and is finite};$$

$$\overline{B}_{c,\lambda}(a) = -\frac{rc}{r - \lambda};$$

$$\overline{B}_{c,\lambda} > -\frac{rc}{r - \lambda}\text{in } (-\infty, a).$$

If $\lambda > r$, the general solution to the ODE will involve sines and cosines, and thus will not satisfy the second line in (3.4). Thus, we may assume that $\lambda \in (0, r)$. The general solution to the homogenous ODE in the first line of (3.4) is of the form $C_1 \exp(\sqrt{\frac{2(r-\lambda)}{D}}\,x) + C_2 \exp(-\sqrt{\frac{2(r-\lambda)}{D}}\,x)$. In light of the requirement in the second line of (3.4), it follows that $\overline{B}_{c,\lambda} = C \exp(\sqrt{\frac{2(r-\lambda)}{D}}\,x)$, for some C. From the third line of (3.4), it follows that

(3.5)
$$\overline{B}_{c,\lambda}(x) = -\frac{rc}{r - \lambda} \exp\big(-\sqrt{\frac{2(r - \lambda)}{D}}\,(a - x)\big).$$

Note that $\overline{B}_{c,\lambda}$ in (3.5) also satisfies the fourth line in (3.4). From (3.3) and (3.5) we obtain

(3.6)
$$B_{c,\lambda}(x) = \frac{rc}{r - \lambda}(1 - \exp\big(-\sqrt{\frac{2(r - \lambda)}{D}}\,(a - x)\big).$$

We now solve for $(c, \lambda) = (c^*, \lambda^*)$ in the equation $B_{c,\lambda}(0) = c$. From (3.6), this equation gives

(3.7)
$$\lambda = r \exp\big(-a\sqrt{\frac{2(r - \lambda)}{D}}\big).$$

It is easy to check that the function $\psi(\lambda) = r \exp\big(-a\sqrt{\frac{2(r-\lambda)}{D}}\big) - \lambda$, is convex for $\lambda \in [0, r]$. It satisfies $\psi(0) > 0$, $\psi(r) = 0$ and $\lim_{\lambda \to r} \psi'(\lambda) = \infty$. Therefore, there exists a unique $\lambda = \lambda^* \in (0, r)$ that solves (3.7). Thus, there exist solutions to (3.1) if $\lambda = \lambda^*$, and thus $\lambda_0(r, 0; a) = \lambda^*$. Up to a positive multiplicative constant, the solution u to (3.1) with $\lambda = \lambda_0(r, a)$ is

given by (3.6) with $\lambda = \lambda_0(r, 0; a)$:

$$(3.8) \quad u(x) = \frac{r}{r - \lambda_0(r, 0; a)} \left(1 - \exp\left(-\sqrt{\frac{2\left(r - \lambda_0(r, 0; a)\right)}{D}}\,(a - x)\right)\right), \; x < a.$$

This proves (1.9) and (1.10). $\qquad\square$

Proof of Corollary 1. From the fact that $\lambda_0(r, 0; a)$ is the unique solution of (3.7) in $(0, r)$, it follows easily that $\lambda_0(r, 0; a)$ is decreasing in a. The corollary follows from this fact along with (3.7). $\qquad\square$

4. Proof of Proposition 4

By linearity, it suffices to prove (1.13) in the case that v, as in the statement of the theorem, is a probability density on $(-\infty, a)$; that is, $v \geq 0$ and $\int_{-\infty}^a v(x)dx = 1$. For such v, we need to show that

$$(4.1) \qquad\qquad \tilde{L}^{(r,0;a)}v(y) = \frac{D}{2}v''(y) - rv(y) + r\delta_0(y).$$

Recall that $P_x^{1;(0)}$ and $E_x^{1;(0)}$ denote probabilities and expectations for the Brownian motion with diffusion parameter D without resetting and started from x. From (2.1), we have

$$(4.2) \quad \begin{aligned} &\frac{1}{t}\left(\tilde{T}_t^{(r,0;a)}v(y) - v(y)\right) = \frac{1}{t}e^{-rt}\left(\int_{-\infty}^a v(x)p^{(a)}(t, x, y) - v(y)\right) + \\ &\frac{1}{t}\int_0^t ds\, re^{-rs}\left(P_v^{1;(0)}(\tau_a > s)\tilde{T}_t^{(r,0;a)}\delta_0(y) - v(y)\right), \end{aligned}$$

where $P_v^{1;(0)}(\tau_a > s) = \int_{-\infty}^a v(x)P_x^{1;(0)}(\tau_a > s)dx$. Clearly,

$$(4.3) \quad \lim_{t \to 0} \frac{1}{t}\int_0^t ds\, re^{-rs}\left(P_v^{1;(0)}(\tau_a > s)\tilde{T}_t^{(r,0;a)}\delta_0(y) - v(y)\right) = r\left(\delta_0(y) - v(y)\right).$$

Also, we have

$$(4.4) \qquad \lim_{t \to \infty} \frac{1}{t}e^{-rt}\left(\int_{-\infty}^a v(x)p^{(a)}(t, x, y) - v(y)\right) = \frac{D}{2}v''(y),$$

by the same argument used for (2.5). The same argument as at the end of the proof of Proposition 3 shows that the convergence in (4.3) and (4.4) is uniform. Thus, (4.1) follows from (4.2)-(4.4).

We now turn to obtaining the principal eigenfunction in (1.14). We need to solve

$$\frac{D}{2}v_0''(y) - (r-\lambda)v_0(y) + r\left(\int_{-\infty}^{a} v_0(x)dx\right)\delta_0(y) = 0;$$

(4.5) $v_0(a) = 0, \quad v_0 > 0 \text{ in } (-\infty, a);$

$$\int_{-\infty}^{a} v_0(y)dy < \infty, \quad \lim_{y\to-\infty} v(y) = 0,$$

where $\lambda = \lambda_0(r, 0; a)$. Let $q = \sqrt{\frac{2(r-\lambda)}{D}}$. Noting that e^{qx} and e^{-qx} are two linearly independent solutions to the linear ODE obtained from the first line in (4.5) by deleting the final term on the left hand side involving the measure δ_0, we look for a solution to (4.5) in the form

(4.6) $$v_0(y) = \begin{cases} e^{qy}, y < 0; \\ ce^{-qy} + (1-c)e^{qy}, \ 0 \le y \le a, \end{cases}$$

for some $c \in \mathbb{R}$. Note that v_0 satisfies the ODE in the first line of (4.5) for $y \neq 0$. Also, v_0 is continuous at $y = 0$ and $\int_{-\infty}^{a} v_0(y)dy < \infty$. In order to obtain $v_0(a) = 0$, we need

(4.7) $$c = \frac{e^{qa}}{e^{qa} - e^{-qa}}.$$

This completely determines v_0 as above, and plugging c from (4.7) into (4.6) shows that $v_0 \geq 0$ and gives (1.14). However, we have not yet dealt with the δ-measure in (4.5). This is where the particular value $\lambda = \lambda_0(r, 0; a)$ comes in. By the Krein-Rutman theory, there must be one (and only one) value of λ for which this v_0 satisfies (4.5). We could stop here, but since the calculations are simple, we now verify this explicitly.

We note that for a continuous function f on $(-\infty, a]$ whose second derivative exists except at $x = 0$ and is bounded near $x = 0$, one has

$$\frac{d}{dy^2}(f(y)) = f''(y) + \left(f'(0^+) - f'(0^-)\right)\delta_0(y),$$

in the sense of distributions. That is,

$$\int_{-\infty}^{a} u''(y)f(y)dy = \int_{-\infty}^{a} u(y)f''(y)dy + (f'(0^+) - f'(0^-))u(0),$$

for smooth u with compact support in $(-\infty, a)$. Thus, writing $v_0(y) = e^{qy} + f(y)$, where $f(y) = 0$, for $y \leq 0$ and $f(y) = c(e^{-qy} - e^{qy})$, for $y \in [0, a]$,

and noting that $f'(0^+) - f'(0^-) = -2cq$, we have

$$(4.8) \qquad \left(\frac{D}{2}v_0''(y) - (r-\lambda)v_0(y)\right) = -Dcq\delta_0(y) = -\frac{Dqe^{qa}}{e^{qa} - e^{-qa}}\delta_0(y).$$

From (4.8), in order that v_0 solve (4.5), we need

$$(4.9) \qquad \int_{-\infty}^{a} v_0(y)dy = \frac{Dq}{r}\frac{e^{qa}}{e^{qa} - e^{-qa}}.$$

A direct calculation reveals that

$$(4.10) \qquad \int_{-\infty}^{a} v_0(y)dy = \frac{2}{q}\frac{e^{qa} - 1}{e^{qa} - e^{-qa}}.$$

Thus, from (4.9) and (4.10) we need

$$(4.11) \qquad \frac{Dqe^{qa}}{r} = \frac{2}{q}(e^{qa} - 1).$$

Recalling that $q = \sqrt{\frac{2(r-\lambda)}{D}}$, (4.11) reduces to $\lambda = re^{-qa} = re^{-a\sqrt{\frac{2(r-\lambda)}{D}}}$. By (1.9) in Theorem 2, it follows that (4.11) holds precisely for $\lambda = \lambda_0(r, 0; a)$. $\square$

5. Proofs of Theorem 3 and Proposition 5

Proof of Theorem 3. We begin with the proof of (1.15). From the standard theory of Markov processes, it follows that

$$f(X(t), t) - \int_0^t (f_t + L^{1;(r,0)}f)(X(s), s)ds$$

is a martingale, for any f satisfying $f \in C_b^{2,1}((-\infty, \infty) \times (0, T)) \cap C((-\infty, \infty) \times [0, \infty))$, for all $T > 0$, where $X(t)$ is the Brownian motion with resetting with generator $L^{1;(r,0)}$ as in (1.1). Then by Doob's optional stopping theorem,

$$f(X(t \wedge \tau_a), t \wedge \tau_a) - \int_0^{t \wedge \tau_a} (f_t + L^{1;(r,0)}f)(X(s), s)ds$$

is also a martingale. Since the process $X(t)$ is stopped at a, we can choose $f(x, t) = e^{\lambda_0(r,0;a)t}u_{r,0;a}(x)$, where $u_{r,0;a}$ is as in (1.10) and solves (3.1) with $\lambda = \lambda_0(r, 0; a)$. This choice of f gives $f_t + L^{1;(r,0)}f = 0$. Thus,

$$e^{\lambda_0(r,0;a)(t \wedge \tau_a)}u_{r,0;a}(X(t \wedge \tau_a)) \text{ is a martingale.}$$

Consequently,

$$(5.1) \qquad E_0^{1;(r,0)}e^{\lambda_0(r,0;a)(t \wedge \tau_a)}u_{r,0;a}(X(t \wedge \tau_a)) = u_{r,0;a}(0).$$

From (1.9) and (1.10), it follows that $u_{r,0;a}(0) = 1$. Also, $u_{r,0;a}$ vanishes at a. Thus, (5.1) reduces to

$$(5.2) \qquad e^{\lambda_0(r,0;a)t} E_0^{1;(r,0)}(u_{r,0;a}(X(t)); \tau_a > t) = 1.$$

Writing $E_0^{1;(r,0)}(u_{r,0;a}(X(t)); \tau_a > t) = P_0^{1;(r,0)}(\tau_a > t) E_0^{1;(r,0)}(u_{r,0;a}(X(t))|\tau_a > t)$, we can rewrite (5.2) in the form

$$(5.3) \qquad P_0^{1;(r,0)}(\tau_a > t) = \frac{1}{E_0^{1;(r,0)}(u_{r,0;a}(X(t))|\tau_a > t)} e^{-\lambda_0(r,0;a)t},$$

which is (1.15).

We now turn to the proof of (1.16). By Proposition 3, the semigroup $T_t^{(r,0;a)}$ is compact. It follows then that

$$(5.4) \qquad \lim_{t \to \infty} E_0^{1;(r,0)}(u_{r,0;a}(X(t))|\tau_a > t) = \frac{\int_{-\infty}^a u_{r,0;a}(x) v_{r,0;a}(x) dx}{\int_{-\infty}^a v_{r,0;a}(x) dx},$$

where $v_{r,0;a}$, appearing in Proposition 4, is the principal eigenfunction corresponding to the principal eigenvalue $\lambda_0(r, 0; a)$ for the adjoint operator $\tilde{L}^{(r,0;a)}$. This follows for example from the corollary after Theorem 3 in [20]. Actually, that corollary, if it could be applied directly to the situation at hand, would give the stronger result that the transition sub-probability density $P_x^{1;(r,0)}(X(t) \in dy | \tau_a > t)$ converges uniformly in x and y to $\frac{v_0(y)}{\int_{-\infty}^a v_0(x) dx}$. However, for the proof of this as in [20], we would need to know that this transition sub-probability density, call it $p^{(r,a)}(t, x, y)$, satisfies $\sup_{x,y \in (-\infty, a)} p^{(r,a)}(1, x, y) < \infty$. The transition probability in [20] satisfied a standard parabolic pde, whereas in the situation at hand $p^{(r,a)}(t, x, y)$ satisfies a nonstandard parabolic pde which includes evaluation at 0. Rather than attempt to prove that the above boundedness condition holds for $p^{(r,a)}(t, x, y)$, we note that in order to prove the weaker form (5.4), the method of proof in [20] works without the necessity of the above uniform pointwise bound.

Letting q be as in (1.16), and recalling the definition of $v_{r,0;a}$ from (1.14), direct calculation reveals that

$$\int_{-\infty}^a v_{r,0;}(x) dx = \frac{2}{q} \frac{e^{qa} - 1}{e^{qa} - e^{-qa}},$$

(as has already been noted in (4.10).) Using (4.11), we can rewrite the right hand side above to obtain

$$(5.5) \qquad \int_{-\infty}^{a} v_{r,0;a}(x)dx = \frac{qDe^{qa}}{r(e^{qa} - e^{-qa})}.$$

Recalling also the definition of $u_{r,0;a}$ from (1.10), a direct calculation gives

$$
(5.6) \quad
\begin{aligned}
\int_{-\infty}^{a} u_{r,0;a}(x)v_{r,0;a}(x)dx &= \frac{r}{r - \lambda_0(r,0;a)}\left(\frac{1}{q} - \frac{e^{-qa}}{2q}\right)+ \\
&\frac{r}{2q(r - \lambda_0(r,0;a))}\frac{1}{e^{qa} - e^{-qa}}\left(2e^{qa} + 2e^{-qa} - e^{-2qa} - 3 - 2qa\right).
\end{aligned}
$$

After some algebra, (1.16) follows from (5.5) and (5.6). Finally, (1.17) follows immediately from (1.15) and (1.16). □

Proof of Proposition 5. Note that for any $y > 0$, $u_{r,0;a}$ satisfies $\lim_{a\to\infty} u_{r,0;a}(x) = 1$, uniformly over $x \in (-\infty, y]$. Thus, to prove (1.18), it suffices to show that the set of distributions $\{P_0^{1;(r,0)}(X(t) \in \cdot | \tau_a > t) : a \geq 1, t > 0\}$ is tight at $+\infty$; namely

$$(5.7) \qquad \lim_{y\to\infty} \sup_{t>0,a\geq 1} P_0^{1;(r,0)}(X(t) \geq y | \tau_a > t) = 0.$$

For each $t > 0$, let LR_t be the random variable denoting the last resetting time before time t for the process $X(t)$ under $P_0^{1;(r,0)}$. Let $\alpha_t(s)$, $0 \leq s \leq t$, denote the density of the random variable $t - LR_t$, and let $\tilde{\alpha}_t(s)$, $0 \leq s \leq t$, denote the density of $t - LR_t$, when conditioned on $\tau_a > t$. Recall that $P_0^{1;(0)}$ denotes probabilities for the Brownian motion without resetting. From the way the resetting mechanism works, we have

$$(5.8) \qquad P_0^{1;(r,0)}(X(t) \geq y | \tau_a > t) = \int_0^t \tilde{\alpha}_t(s) P_0^{1;(0)}(X(s) \geq y | \tau_a > s)ds.$$

We now show that

$$(5.9) \qquad P_0^{1;(0)}(X(s) \geq y | \tau_a > s) \leq P_0^{1;(0)}(X(s) \geq y).$$

Under $P_0^{1;(0)}$, the process $X(u), 0 \leq u \leq s$, conditioned on $\tau_a > s$, is a time-inhomogeneous diffusion process generated by $\frac{D}{2}\left(\frac{d^2}{dx^2} + b^{(s)}(u,x)\frac{d}{dx}\right)$, where $b^{(s)}(u,x) = \frac{w_x(s-u,x)}{w(s-u,x)}$ with $w(u,x) = P_x^{1;(0)}(\tau_a > u)$, $x < a$ [19]. Clearly, $w_x(s-u,x) \leq 0$. Thus, the drift $b^{(s)}$ is non-positive. Now (5.9) follows from this along with the Ikeda-Watanabe comparison theorem [16].

From the definitions of α_t and $\tilde{\alpha}_t$, we have

$$(5.10) \qquad \tilde{\alpha}_t(s) = \frac{P_0^{1;(0)}(\tau_a > t - s)P_0^{1;(0)}(\tau_a > s)}{P_0^{1;(0)}(\tau_a > t)}\alpha_t(s).$$

We have

$$(5.11) \qquad \frac{P_0^{1;(0)}(\tau_a > t - s)P_0^{1;(0)}(\tau_a > s)}{P_0^{1;(0)}(\tau_a > t)} \leq \frac{P_0^{1;(0)}(\tau_a > \frac{t}{2})}{P_0^{1;(0)}(\tau_a > t)}.$$

As is well-known from the reflection principle,

$$P_0^{1;(0)}(\tau_a > u) = 2\int_0^{\frac{a}{\sqrt{Du}}} \frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2})dz, \ a > 0.$$

Thus, the right hand side of (5.11) is bounded in t. Using this with (5.8)-(5.11), we have

$$(5.12) \qquad P_0^{1;(r,0)}(X(t) \geq y|\tau_a > t) \leq C\int_0^t P_0^{1;(0)}(X(s) \geq y)\alpha_t(s)ds,$$

for some $C > 0$. Thus to prove (5.7), it suffices to show that

$$(5.13) \qquad \lim_{y\to\infty}\sup_{t>0}\int_0^\infty \alpha_t(s)P_0^{1;(0)}(X(s) \geq y)ds = 0.$$

Clearly

$$(5.14) \qquad \lim_{y\to\infty} P_0^{1;(0)}(X(s) \geq y) = 0, \ \text{uniformly over } s \text{ in a compact set.}$$

The distribution with density α_t is obviously stochastically dominated by the time that elapses between the largest resetting time smaller than t and the smallest resetting time larger than t. This distribution is well-known; it has density

$$f_t(s) = \begin{cases} r^2se^{-rs}, 0 \leq s \leq t; \\ r(1 + rt)e^{-rs}, s > t. \end{cases}$$

[14, p.13]. Clearly the set of densities $\{f_t\}_{0<t<\infty}$ is tight. Thus, the set of densities $\{\alpha_t\}_{0<t<\infty}$ is tight. From this and (5.14), it follows that (5.13) holds. $\qquad\qquad \square$

6. Proof of Proposition 2-Bessel

The proof follows similarly to the proof of Proposition 3. Let $\mathcal{P}_x^{\mathrm{Bes}(d);0}$ and $\mathcal{E}_x^{\mathrm{Bes}(d);0}$ denote probabilities and expectations for the Bessel process of order d and diffusion coefficient D without resetting and starting from x. Similar to (2.1), we have

(6.1)
$$\mathcal{T}_t^{(r,A;\epsilon_0)} f(x) = e^{-rt} \mathcal{E}_x^{\mathrm{Bes}(d);0}(f(Y(t); \tau_{\epsilon_0}^{(Y)} > t) +$$
$$\int_0^t ds\, r e^{-rs} \mathcal{P}_x^{\mathrm{Bes}(d);0}(\tau_{\epsilon_0}^{(Y)} > s) \mathcal{T}_{t-s}^{(r,A;\epsilon_0)} f(0).$$

Using (6.1), the proof that the semigroup operator $\mathcal{T}_t^{(r,A;\epsilon_0)}$ maps $C_{0_{\epsilon_0}}([\epsilon_0, \infty])$ to $C_{0_{\epsilon_0}}([\epsilon_0, \infty])$ is just like the corresponding proof in Proposition 3, because the basic properties of the Brownian motion semigroup that were used in the proof are shared by the Bessel process semigroup. This is also true with regard to the calculation of the generator.

With regard to the proof that the operator is compact, we rewrite (6.1) as

(6.2)
$$\mathcal{T}_t^{(r,A;\epsilon_0)} f(x) = e^{-rt} \int_{\epsilon_0}^\infty p^{\mathrm{Bes}(d);\epsilon_0}(t,x,y) f(y) dy +$$
$$\int_0^t ds\, r e^{-rs} \mathcal{P}_x^{\mathrm{Bes}(d);0}(\tau_{\epsilon_0}^{(Y)} > s) \mathcal{T}_{t-s}^{(r,A;\epsilon_0)} f(0),$$

where $p^{\mathrm{Bes}(d);\epsilon_0}(t,x,y)$ denotes the transition sub-probability density for the Bessel process without resetting and killed upon hitting ϵ_0. From (6.2), it suffices to show that the functions $\{p^{\mathrm{Bes}(d);\epsilon_0}(t,\cdot,y)\}_{y \in (\epsilon_0, \infty)}$ and the functions $\{\mathcal{P}_\cdot^{\mathrm{Bes}(d);0}(\tau_{\epsilon_0}^{(Y)} > s)\}_{s>0}$ are uniformly equicontinuous on (ϵ_0, ∞). We sketch how this equicontinuity can be deduced from [4] and [5].

In those two papers, the parameter μ plays the role of our $\frac{d}{2} - 1$ (or equivalently, $2\mu + 2$ plays the role of our d), and in [4] the parameter a plays the role of our ϵ_0. Also, our D is equal to 1 in those papers. So we only consider this case. (The general case follows by scaling.) In [4], $\{p_a^\mu(t,\cdot,y)\}_{y \in (a,\infty)}$ plays the role of our $\{p^{\mathrm{Bes}(d),\epsilon_0}(t,\cdot,y)\}_{y \in (\epsilon_0,\infty)}$ and $\{q_{\cdot,a}^\mu(s)\}_{s>0}$ plays the role of our $\{\mathcal{P}_\cdot^{\mathrm{Bes}(d);0}(\tau_{\epsilon_0}^{(Y)} > s)\}_{s>0}$. So we need to demonstrate the uniform equicontinuity of $\{p_a^\mu(t,\cdot,y)\}_{y \in (a,\infty)}$ and of $\{q_{\cdot,a}^\mu(s)\}_{s>0}$ over (ϵ_0, ∞).

From [4, (2.10)], the uniform equicontinuity of $\{p_a^\mu(t,\cdot,y)\}_{y \in (a,\infty)}$ follows from that of $\{q_{\cdot,a}^\mu(s)\}_{s>0}$ and of $\{p^\mu(t,\cdot,y)\}_{y \in (a,\infty)}$, where $p^\mu(t,x,y)$ is the

transition probability function for the Bessel process of order $d = 2\mu+2$ without killing. Using [4, equations (1.1) and (2.3)] shows the uniform equicontinuity of $\{p^\mu(t, \cdot, y)\}_{y \in (a, \infty)}$. The uniform equicontinuity of $\{q^\mu_{\cdot, a}(s)\}_{s > 0}$ can be deduced from section 2.3 in [5]. □

7. Proof of Theorem 2-Bessel and Corollary 1-Bessel

Proof of Theorem 2-Bessel. The proof follows the contours of the proof of Theorem 2, except that instead of using the operator $\frac{D}{2}\frac{d^2}{dx^2}$ on $(-\infty, a]$, we use the operator $\frac{D}{2}\frac{d^2}{dx^2} + \frac{D(d-1)}{2x}\frac{d}{dx}$ on $[\epsilon_0, \infty)$, and instead of the distinguished point being 0, it is A. We use the same notation u, $B_{c,\lambda}$ and $\overline{B}_{c,\lambda}$ for the functions appearing in that proof. As noted after Proposition 2-Bessel, $\mathcal{L}^{(r, A; \epsilon_0)}$ has a compact resolvent. Thus, by Proposition 2-Bessel and the Krein-Rutman theorem, it follows that the principal eigenvalue $\lambda_0(r, A; \epsilon_0)$ is the unique solution λ to the following equation, analogous to (3.1):

$$\frac{D}{2}u''(x) + D\frac{(d-1)}{2x}u'(x) + r(u(A) - u(x)) = -\lambda u \ \text{ in } (\epsilon_0, \infty);$$

(7.1) $\quad \lim_{x \to \infty} u(x)$ exists and is finite;

$$u(\epsilon_0) = 0;$$

$$u > 0 \text{ in } (\epsilon_0, \infty).$$

And the function $\overline{B}_{c,\lambda}$ satisfies the following equation, analogous to (3.4):

$$\frac{D}{2}\overline{B}''_{c,\lambda} + D\frac{(d-1)}{2x}\overline{B}'_{c,\lambda} + (\lambda - r)\overline{B}_{c,\lambda} = 0, \ x \in (\epsilon_0, \infty);$$

(7.2) $\quad \lim_{x \to \infty} \overline{B}_{c,\lambda}(x)$ exists and is finite;

$$\overline{B}_{c,\lambda}(\epsilon_0) = -\frac{rc}{r - \lambda};$$

$$\overline{B}_{c,\lambda} > -\frac{rc}{r - \lambda} \text{ in } (\epsilon_0, \infty).$$

The modified Bessel functions of the first and second kind, of order ν, denoted respectively by I_ν and K_ν are linearly independent solutions to the linear ODE $x^2\frac{d^2W}{dx^2} + x\frac{dW}{dx} - (x^2 + \nu^2)W = 0$ [1, 26]. By looking for solutions of the form $x^\gamma K_\nu(\eta x)$ and $x^\gamma I_\nu(\eta x)$, for parameters γ, ν and η, one can verify that two linearly independent solutions to the ODE in (7.2) are:

$$x^{\frac{2-d}{2}}K_{\frac{d-2}{2}}\left(\sqrt{\frac{2}{D}(r - \lambda)}\, x\right), \quad x^{\frac{2-d}{2}}I_{\frac{d-2}{2}}\left(\sqrt{\frac{2}{D}(r - \lambda)}\, x\right).$$

The function I_ν grows exponentially and the function K_ν decays exponentially as $x \to \infty$ [1, 26]. Therefore, it follows from (7.2) that

$$\overline{B}_{c,\lambda} = -\frac{rc}{r-\lambda} \frac{\epsilon_0^{\frac{d-2}{2}}}{K_{\frac{d-2}{2}}(\sqrt{\frac{2}{D}(r-\lambda)}\,\epsilon_0)} x^{\frac{2-d}{2}} K_{\frac{d-2}{2}}(\sqrt{\frac{2}{D}(r-\lambda)}\,x).$$

Similar to the passage from (3.5) to (3.6), we have

$$(7.3) \qquad B_{c,\lambda}(x) = \frac{rc}{r-\lambda}\Big(1 - \big(\frac{x}{\epsilon_0}\big)^{\frac{2-d}{2}} \frac{K_{\frac{d-2}{2}}(\sqrt{(r-\lambda)\frac{2}{D}}\,x)}{K_{\frac{d-2}{2}}(\sqrt{(r-\lambda)\frac{2}{D}}\,\epsilon_0)}\Big)$$

Similar to the proof of Theorem 2, we now solve for $(c, \lambda) = (c^*, \lambda^*)$ in the equation $B_{c,\lambda}(A) = c$. From (7.3), this yields the equation (1.23) for $\lambda = \lambda^*$, with $c = c^*$ being arbitrary. Although we could perform an analysis to show directly that there exists a unique $\lambda^* \in (0, r)$ that solves (1.23), similar to what was done in the proof of Theorem 2, but much more tedious, in the present case we simply note that this follows by the uniqueness of the principal eigenvalue in the Krein-Rutman theorem. This unique solution is the principal eigenvalue $\lambda_0(r, A; \epsilon_0)$. A principal eigenfunction $\mathcal{U}_{0;A}(x)$ is then given by the right hand side of (7.3) with $\lambda = \lambda_0(r, A; \epsilon_0)$ and, say, $c = 1$. □

Proof of Corollary 1-Bessel. The corollary follows readily from (1.23) and the asymptotic estimate $K_{\frac{d-2}{2}}(A) \sim \sqrt{\frac{\pi}{2A}} e^{-A}$ as $A \to \infty$ [1, 26]. (Note that this leading order asymptotic behavior for $K_{\frac{d-2}{2}}(A)$ is independent of the order $\frac{d-2}{2}$.) □

8. Proof of Proposition 3-Bessel

The proof that the adjoint generator is as in (1.27) is similar to the proof of Proposition 4, so we leave it to the reader. We turn to the calculation of the corresponding principal eigenfunction in (1.28). We need to solve

(8.1)
$$\frac{D}{2}v''(y) - D\frac{d-1}{2x}v' + D\frac{d-1}{2x^2}v - (r-\lambda)v(y) + r\Big(\int_{\epsilon_0}^\infty v(x)dx\Big)\delta_A(y) = 0;$$

$$v(\epsilon_0) = 0, v > 0 \text{ in } (\epsilon_0, \infty);$$

$$\int_{\epsilon_0}^\infty v(y)dy < \infty, \ \lim_{y\to\infty} v(y) = 0,$$

where $\lambda = \lambda(r, A; \epsilon_0)$. From the proof of Theorem 2-Bessel, recall K_ν and I_ν, the modified Bessel functions of order ν, which are linearly independent solutions to the linear ODE $x^2 \frac{d^2W}{dx^2} + x \frac{dW}{dx} - (x^2 + \nu^2)W = 0$. By looking for solutions of the form $x^\gamma K_\nu(\eta x)$, for parameters γ, ν and η, one can verify that two linearly independent solutions to the linear ODE obtained from the first line of (8.1) by deleting the final term involving the measure δ_A are

$$x^{\frac{d}{2}} K_{\frac{d-2}{2}}\left(\sqrt{\frac{2}{D}(r - \lambda)}\, x\right), \qquad x^{\frac{d}{2}} I_{\frac{d-2}{2}}\left(\sqrt{\frac{2}{D}(r - \lambda)}\, x\right).$$

Recalling that $I_{\frac{d-2}{2}}$ grows exponentially and $K_{\frac{d-2}{2}}$ decays exponentially, we look for the solution v to (8.1) in the form

$$(8.2) \quad v(x) = \begin{cases} c_1 x^{\frac{d}{2}} I_{\frac{d-2}{2}}\left(\sqrt{\frac{2}{D}(r - \lambda)}\, x\right) + c_2 x^{\frac{d}{2}} K_{\frac{d-2}{2}}\left(\sqrt{\frac{2}{D}(r - \lambda)}\, x\right), \\ \epsilon_0 \leq x \leq A; \\ x^{\frac{d}{2}} K_{\frac{d-2}{2}}\left(\sqrt{\frac{2}{D}(r - \lambda)}\, x\right), \ x \geq A. \end{cases}$$

Then v satisfies the third line of (8.1) and it satisfies the ODE in the first line of (8.1) for $y \neq A$. Using the equations $v(\epsilon_0) = 0$ and $v(A^-) = v(A^+)$, we can solve for c_1 and c_2, obtaining (1.28). The δ-measure requirement at $y = A$ in the first line of (8.1) follows automatically from the Krein-Rutman theorem in the case that $\lambda = \lambda(r, A; \epsilon_0)$. (See the discussion at the corresponding juncture of the proof of Proposition 4, which contains the corresponding result in the one-dimensional case.)

9. Proofs of Theorem 3-Bessel and Proposition 4-Bessel

Proof of Theorem 3-Bessel. The proof is just like the proof of Theorem 3. $\square$

Proof of Proposition 4-Bessel. From (1.24) we have $\lim_{x,A \to \infty} \mathcal{U}_{r,A;\epsilon}(x) = 1$. Thus, to prove (1.32), it suffices to prove that

$$(9.1) \qquad \lim_{y \to \infty} \limsup_{A \to \infty} \sup_{t > 0} \mathcal{P}_A^{(r;A)}(Y(t) \leq A - y | \tau_{\epsilon_0}^{(Y)} > t) = 0.$$

For each $t > 0$, let LR_t be the random variable denoting the last resetting time before t for the process $Y(\cdot)$ under $\mathcal{P}_A^{(r;A)}$. Let $\alpha_t(s), 0 \leq s \leq t$, denote the density of the random variable $t - \mathrm{LR}_t$, and let $\tilde{\alpha}_t(s), 0 \leq s \leq t$, denote the density of $t - \mathrm{LR}_t$ when conditioned on $\tau_{\epsilon_0}^{(Y)} > t$. From the way the

resetting mechanism works, we have
(9.2)
$$\mathcal{P}_A^{(r;A)}(Y(t) \le A - y | \tau_{\epsilon_0}^{(Y)} > t) = \int_0^t \tilde{\alpha}_t(s) \mathcal{P}_A^{(r;A)}(Y(s) \le A - y | \tau_{\epsilon_0}^{(Y)} > s) ds.$$

We now show that

$$(9.3) \qquad \mathcal{P}_A^{(r;A)}(Y(s) \le A - y | \tau_{\epsilon_0}^{(Y)} > s) \le \mathcal{P}_A^{(r;A)}(Y(s) \le A - y).$$

Under $\mathcal{P}_A^{(r;A)}$, the process $Y(u), 0 \le u \le s$, conditioned on $\tau_{\epsilon_0}^{(Y)} > s$, is a time-inhomogeneous diffusion process generated by $\frac{D}{2}(\frac{d^2}{dy^2} + \frac{d-1}{y}\frac{d}{dy} + b^{(s)}(u,y)\frac{d}{dy})$, where $b^{(s)}(u,y) = \frac{w_y(s-u,y)}{w(s-u,y)}$ with $w(u,y) = \mathcal{P}_y^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > u)$ [19]. Clearly, $w_y(s-u,y) \ge 0$. Thus, the drift $b^{(s)}$ is nonnegative. Now (9.3) follows from this along with the Ikeda-Watanabe comparison theorem [16].

From the definitions of α_t and $\tilde{\alpha}_t$, we have

$$(9.4) \qquad \tilde{\alpha}_t(s) = \frac{\mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > t - s)) \mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > s)}{\mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > t)} \alpha_t(s).$$

We have

$$(9.5) \qquad \frac{\mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > t - s)) \mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > s)}{\mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > t)} \le \frac{\mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > \frac{t}{2})}{\mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > t)}.$$

For $d \ge 3$, the Bessel process of order d is transient, so $\lim_{t\to\infty} \mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > t) > 0$, and thus the right hand side of (9.5) is bounded in t. For $d = 2$, $\mathcal{P}_A^{(r;A)}(\tau_{\epsilon_0}^{(Y)} > t)$ has logarithmic decay ([21, p.224]) from which it follows that the right hand side of (9.5) is bounded in t. Using this with (9.2)-(9.5), we have

$$(9.6) \quad \mathcal{P}_A^{(r;A)}(Y(t) \le A - y | \tau_{\epsilon_0}^{(Y)} > t) \le C \int_0^t \mathcal{P}_A^{(r;A)}(Y(s) \le A - y) \alpha_t(s) ds,$$

for some $C > 0$. Since $Y(\cdot)$ is a Bessel process of order d, it is clear that

$$(9.7) \qquad \lim_{y\to\infty} \limsup_{A\to\infty} \mathcal{P}_A^{(r;A)}(Y(s) \le A - y) = 0, \text{ for all } s > 0.$$

As shown at the end of the proof of Proposition 5, the distributions $\{\alpha_t\}_{0 \le t < \infty}$ are tight. Now (9.1) follows from this along with (9.6) and (9.7). $\qquad\square$

10. Proof of Theorem 1

We first prove the one-dimensional case. In light of (1.6), it suffices to prove (1.7) with the range of integration from 0 to ∞ instead of from $-\infty$ to ∞. From (1.6) and (1.15), we have

(10.1)
$$\int_0^\infty P_0^{1;(r,0)}(\tau_a > t)\mu_{B,l}(da) = \int_0^\infty \frac{e^{-\lambda_0(r,0;a)\,t}}{E_0^{1;(r,0)}(u_{r,0;a}(X(t))|\tau_a > t)}c(a)e^{-Ba^l}da,$$

as $t \to \infty$,

where

(10.2)
$$\lim_{a\to\infty} \frac{\log c(a)}{a^l} = 0.$$

Since $\lambda_0(r,0;a)$ is decreasing to 0 as $a \to \infty$, it is clear that the asymptotic behavior of the right hand side of (10.1) as $t \to \infty$ depends only on large a. Thus, in light of (1.18),

(10.3)
$$\int_0^\infty \frac{e^{-\lambda_0(r,0;a)\,t}}{E_0^{1;(r,0)}(u_{r,0;a}(X(t))|\tau_a > t)}c(a)e^{-Ba^l}da \sim \int_0^\infty c(a)e^{-\lambda_0(r,0;a)\,t-Ba^l}da, \text{ as } t \to \infty.$$

By (1.9), we can replace $\lambda_0(r,0;a)$ in the exponent on the right hand side of (10.3) by $re^{-\sqrt{\frac{2(r-\lambda_0(r,0;a))}{D}}\,a}$. Making this replacement, using the fact that $\lambda(r,0;a)$ approaches 0 as $a \to \infty$, and using (10.2), it follows that if

(10.4)
$$\lim_{t\to\infty} \frac{1}{(\log t)^l} \log \int_0^\infty \exp(-Rte^{-\kappa a} - Ba^l)da = -\frac{B}{\kappa^l}, \text{ for all } B, R, \kappa > 0,$$

then

(10.5)
$$\lim_{t\to\infty} \frac{1}{(\log t)^l} \log \int_0^\infty c(a)e^{-\lambda_0(r,0;a)t-Ba^l}da = -B(\frac{D}{2r})^{\frac{l}{2}}.$$

(In fact, it is unnecessary here to replace the specific r with the generic R; however, we will need this general form of (10.4) in the proof of the multidimensional case.) Therefore, from (10.1), (10.3) and (10.5), it follows that the proof of (1.7) will be completed if we prove (10.4).

To analyze the left hand side of (10.4), we locate, for each large t, the minimum of the expression

(10.6)
$$\gamma_t(a) := Rte^{-\kappa a} + Ba^l.$$

First consider the case that $l \geq 1$. In this case, γ_t is convex and $\gamma_t'(0) < 0$, for all sufficiently large t (actually all t, if $l > 1$). Thus, for large t, it has a unique minimum which occurs at some a^* which satisfies

$$(10.7) \qquad \kappa R t e^{-\kappa a^*} = l B (a^*)^{l-1}.$$

Substituting from (10.7), we have

$$(10.8) \qquad \gamma_t(a^*) = R t e^{-\kappa a^*} + B(a^*)^l = B(a^*)^{l-1}\Big(\frac{l}{\kappa} + a^*\Big).$$

From (10.7) we have

$$\log(\kappa R) + \log t - \kappa a^* = \log(l B) + (l-1)\log a*,$$

from which it follows that

$$(10.9) \qquad a^* \sim \frac{1}{\kappa}\log t, \ \text{ as } t \to \infty.$$

Substituting from (10.9) into the right hand side of (10.8), we have

$$(10.10) \qquad \gamma_t(a^*) \sim B(a^*)^l = B\Big(\frac{\log t}{\kappa}\Big)^l, \ \text{ as } t \to \infty.$$

Now consider the case $l \in (0,1)$. We have $\gamma_t'(a) = -\kappa R t e^{-\kappa a} + l B a^{l-1}$. Note that $\gamma_t'(0^+) = \infty$, and it is easy to see that for each t, $\gamma_t'(a) > 0$ for sufficiently large a. However, $\gamma_t'(1) < 0$, for sufficiently large t. Thus, for sufficiently large t, there must be at least two roots to $\gamma_t'(a) = 0$. Substituting $\kappa R t e^{-\kappa a} = l B a^{l-a}$ into $\gamma_t''(a)$, it follows that if $\gamma_t'(a) = 0$, then $\gamma_t''(a) = B l a^{l-2}(\kappa a + l - 1)$. Thus, a zero a of γ_t' is a relative maximum of γ_t if $a < \frac{1-l}{\kappa}$ and is a relative minimum if $a > \frac{1-l}{\kappa}$. From this it follows that for sufficiently large t, there are exactly two zeroes of γ_t', and that the larger one is the global minimum of γ_t, Denote this global minimum by a^*. Using the fact that $a^* > \frac{1-l}{\kappa}$ and that $0 = \gamma_t'(a^*) = -\kappa R t e^{-\kappa a^*} + l B (a^*)^{l-1}$, it follows that a^* approaches ∞ as $t \to \infty$. The rest of the analysis is as before.

Thus, for sufficiently large t, (10.9) and (10.10) hold for all $l > 0$, and from the previous paragraph, for all $l > 0$ we have

$$(10.11) \qquad \gamma_t''(a^*) = B l (a^*)^{l-2}(\kappa a^* + l - 1).$$

Note that

$$(10.12) \qquad \gamma_t''(a) = \kappa^2 R t e^{-\kappa a} + l(l-1) B a^{l-2}.$$

Using (10.11) and (10.12), we have

$$\gamma_t''(a) = \gamma_t''(a^*) + (\gamma_t''(a) - \gamma_t''(a^*)) = Bl(a^*)^{l-2}\big(\kappa a^* + l - 1\big) +$$

(10.13) $$\kappa^2 Rt(e^{-\kappa a} - e^{-\kappa a^*}) + l(l-1)B(a^{l-2} - (a^*)^{l-2}) \le$$

$$Bl(a^*)^{l-2}\big(\kappa a^* + l - 1\big) + l(l-1)B(a^{l-2} - (a^*)^{l-2}), \text{ for } a > a^*.$$

From (10.13) and (10.9) it follows that for some constant $C > 0$,

(10.14) $$\gamma''(a) \le C \log^{l-1} t, \text{ for } a \in [a^*, a^* + 1].$$

Since $\gamma_t'(a^*) = 0$, it follows from (10.14) that

(10.15) $$\gamma_t(a) \le \gamma_t(a^*) + \frac{1}{2}C(\log^{l-1} t)(a - a^*)^2, \text{ for } a \in [a^*, a^* + 1].$$

Thus, we conclude from (10.15) that for some $\alpha > 0$,

(10.16) $$\gamma_t(a) \le \gamma_t(a^*) + 1, \text{ for } \begin{cases} a \in [a^*, a^* + \alpha], \text{ if } l \in (0,1]; \\ a \in \big[a^*, a^* + \frac{\alpha}{\log^{l-1} t}\big], \text{ if } l > 1. \end{cases}$$

Note that the two cases of the interval appearing on the right hand side of (10.16) can be merged by writing $\big[a^*, a^* + \frac{\alpha}{(\log t)^{\max(0, l-1)}}\big]$. From this observation along with (10.16), (10.10) and the definition of γ_t in (10.6), we obtain the lower bound

(10.17)
$$\int_0^\infty \exp(-Rte^{-\kappa a} - Ba^l)da \ge \frac{\alpha}{(\log t)^{\max(0, l-1)}} \exp\big(-(1+\epsilon)B(\frac{\log t}{\kappa})^l - 1\big),$$

for any $\epsilon > 0$ and for sufficiently large t depending on ϵ.

Now we turn to an upper bound for the left hand side of (10.17). Applying L'Hôpital's rule to $\frac{\int_x^\infty e^{-Ba^l} da}{x^{-l+1}e^{-Bx^l}}$ shows that

$$\int_x^\infty e^{-Ba^l} da \sim \frac{x^{-l+1}e^{-Bx^l}}{lB}, \text{ as } x \to \infty,$$

and thus,

(10.18)
$$\int_x^\infty e^{-Ba^l} da \le \frac{x^{-l+1}e^{-(1-\epsilon)Bx^l}}{lB}, \text{ for any } \epsilon > 0 \text{ and sufficiently large } x \text{ depending on } \epsilon.$$

Write

(10.19)
$$\int_0^\infty \exp(-Rte^{-\kappa a} - Ba^l)da =$$
$$\int_0^{a^*} \exp(-Rte^{-\kappa a} - Ba^l)da + \int_{a^*}^\infty \exp(-Rte^{-\kappa a} - Ba^l)da.$$

Using (10.18) and (10.9) gives

(10.20)
$$\int_{a^*}^{\infty} \exp(-Rte^{-\kappa a} - Ba^l)da \leq \int_{a^*}^{\infty} e^{-Ba^l}da \leq \frac{1}{lB}\left(\frac{\log t}{\kappa}\right)^{-l+1} e^{-(1-\epsilon)B\left(\frac{\log t}{\kappa}\right)^l},$$

for any $\epsilon > 0$ and sufficiently large t depending on ϵ.

From the definition of γ_t in (10.6) and the fact that a^* is the minimum of $\gamma_t(a)$, it follows from (10.9) and (10.10) that

(10.21)
$$\int_0^{a^*} \exp(-Rte^{-\kappa a} - Ba^l)da \leq (1+\epsilon)\frac{\log t}{\kappa}e^{-(1-\epsilon)B\left(\frac{\log t}{\kappa}\right)^l},$$

for any $\epsilon > 0$ and sufficiently large t depending on ϵ.

From (10.19)-(10.21), we conclude that

(10.22)
$$\int_0^{\infty} \exp(-Rte^{-\kappa a} - Ba^l)da \leq$$
$$(1+\epsilon)\frac{\log t}{\kappa}e^{-(1-\epsilon)B\left(\frac{\log t}{\kappa}\right)^l} + \frac{1}{lB}\left(\frac{\log t}{\kappa}\right)^{-l+1} e^{-(1-\epsilon)B\left(\frac{\log t}{\kappa}\right)^l},$$

for any $\epsilon > 0$ and sufficiently large t depending on ϵ.

Now (10.4) follows from (10.17) and (10.22). This completes the proof of the one-dimensional case.

We now turn to the multi-dimensional case, where we will also utilize (10.4). For $A > 0$, let $\bar{\mu}_{B,l}^{(d)}(A) = \int_{|x|=1} \mu_{B,l}^{(d)}(Ax)s_d(dx)$, where s_d denotes Lebesgue measure on the unit sphere in R^d. We have

(10.23)
$$\int_{\mathbb{R}^d} P_0^{d;(r,0)}(\tau_a > t)\mu_{B,l}^{(d)}(da) = \int_{\epsilon_0}^{\infty} \mathcal{P}_{|a|}^{(r,|a|)}(\tau_{\epsilon_0}^{(Y)} > t)A^{d-1}\bar{\mu}_{B,l}^{(d)}(A)dA =$$
$$\int_{\epsilon_0}^{\infty} \frac{1}{\mathcal{E}_A^{(r,A)}(\mathcal{U}_{r,A;\epsilon_0}(Y(t))|\tau_{\epsilon_0}^{(Y)} > t)}e^{-\lambda_0(r,A;\epsilon_0)\,t}A^{d-1}\bar{\mu}_{B,l}^{(d)}(A)dA \sim$$
$$\int_{\epsilon_0}^{\infty} \frac{1}{\mathcal{E}_A^{(r,A)}(\mathcal{U}_{r,A;\epsilon_0}(Y(t))|\tau_{\epsilon_0}^{(Y)} > t)}e^{-\lambda_0(r,A;\epsilon_0)\,t}C(A)e^{-BA^l}dA,$$

where

(10.24)
$$\lim_{A\to\infty} \frac{\log C(A)}{A^l} = 0.$$

The first equality in (10.23) follows from (1.19) and (1.20), the second one follows from Theorem 3-Bessel, and the third one follows from (1.6). Since $\lambda_0(r, A; \epsilon_0)$ is decreasing to 0 as $A \to \infty$, it is clear that the asymptotic

behavior of the right hand side of (10.23) as $t \to \infty$ depends only on large A. Thus, in light of (1.32),

(10.25)
$$\int_{\epsilon_0}^{\infty} \frac{1}{\mathcal{E}_A^{(r,A)}(\mathcal{U}_{r,A;\epsilon_0}(Y(t))|\tau_{\epsilon_0}^{(Y)} > t)} e^{-\lambda_0(r,A;\epsilon_0)\, t} C(A) e^{-BA^l} dA \sim$$
$$\int_{\epsilon_0}^{\infty} C(A) e^{-\lambda_0(r,A;\epsilon_0)\, t - BA^l} dA, \text{ as } t \to \infty.$$

Using (1.23) and the fact that $K_{\frac{d-2}{2}}(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ as $x \to \infty$ [1, 26], we have

(10.26)
$$\lambda_0(r, A; \epsilon_0) \sim \eta\, A^{\frac{1-d}{2}} e^{-\sqrt{(r-\lambda_0(r,A;\epsilon_0))\frac{2}{D}}\, A}, \text{ as } A \to \infty, \text{ for some } \eta > 0.$$

Using (10.26) and the fact that the $t \to \infty$ asymptotic behavior depends only on large A, we have for any $\delta > 0$,

(10.27)
$$\int_{\epsilon_0}^{\infty} C(A) \exp(-t(\eta + \delta) A^{\frac{1-d}{2}} e^{-\sqrt{(r-\lambda_0(r,A;\epsilon_0))\frac{2}{D}}\, A} - BA^l) dA \le$$
$$\int_{\epsilon_0}^{\infty} C(A) e^{-\lambda_0(r,A;\epsilon_0)\, t - BA^l} dA \le$$
$$\int_{\epsilon_0}^{\infty} C(A) \exp(-t(\eta - \delta) A^{\frac{1-d}{2}} e^{-\sqrt{(r-\lambda_0(r,A;\epsilon_0))\frac{2}{D}}\, A} - BA^l) dA,$$
for sufficiently large t.

By (10.24) and the fact that $\lim_{A \to \infty} \lambda_0(r, A; \epsilon_0) = 0$, it follows that for any $\delta > 0$, we have for sufficiently large A,

(10.28)
$$\exp(-t(\eta \pm \delta) e^{-\sqrt{(\frac{2r}{D} - \delta)A}} - (B + \delta) A^l) \le$$
$$C(A) \exp(-t(\eta \pm \delta) A^{\frac{1-d}{2}} e^{-\sqrt{(r-\lambda_0(r,A;\epsilon_0))\frac{2}{D}}\, A} - BA^l) \le$$
$$\exp(-t(\eta \pm \delta) e^{-\sqrt{(\frac{2r}{D} + \delta)A}} - (B - \delta) A^l).$$

Applying (10.4) to the integrals $\int_{\epsilon}^{\infty} \exp(-t(\eta \pm \delta) e^{-\sqrt{(\frac{2r}{D} - \delta)A}} - (B + \delta) A^l) dA$ and $\int_{\epsilon}^{\infty} \exp(-t(\eta \pm \delta) e^{-\sqrt{(\frac{2r}{D} + \delta)A}} - (B - \delta) A^l) dA$, the proof of (1.7) now follows from (10.23), (10.25), (10.27) and (10.28). $\square$

11. Proof of Proposition 1

Recall that $P_x^{2;(0)}$ denotes probabilities for the standard two-dimensional Brownian motion starting from $x \in \mathbb{R}^2$. By symmetry, one has

$$(11.1) \qquad P_0^{2;(0)}(\tau_a > t) = P_a^{2;(0)}(\tau_0 > t).$$

This latter probability satisfies

$$(11.2) \qquad P_a^{2;(0)}(\tau_0 > t) \approx 1 \wedge \frac{2 \log \frac{|a|}{\epsilon_0}}{\log t}, \text{ for } t \geq 2 \text{ and } |a| > \epsilon_0,$$

where $f(a,t) \approx g(a,t)$ means that there are constants $c_1, c_2 > 0$, independent of a and t, such that $c_1 f(a,t) \leq g(a,t) \leq c_2 f(a,t)$. This follows from the formula eight lines up from the bottom on page 774 of [5]. The form there is slightly different from (11.2), but is equivalent. From (11.1) and (11.2), it follows that

$$(11.3) \qquad \begin{aligned} &\lim_{t\to\infty} P_0^{2;(0)}(\tau_{a_t} > t) = 0, \text{ if } \lim_{t\to\infty} \frac{|a_t|}{t^\delta} = 0, \text{ for all } \delta > 0; \\ &\liminf_{t\to\infty} P_0^{2;(0)}(\tau_{a_t} > t) > 0, \text{ if } \lim_{t\to\infty} \frac{|a_t|}{t^\delta} > 0, \text{ for some } \delta > 0. \end{aligned}$$

By Brownian scaling,

$$(11.4) \qquad \lim_{t\to\infty} P_0^{2;(0)}(\max_{0\leq s\leq t} |X(t)| \geq |a_t|) = 0, \text{ if } \lim_{t\to\infty} \frac{|a_t|}{t^{\frac{1}{2}}} = \infty.$$

Now (1.3) follows from (11.3) and (11.4). $\qquad\qquad\square$

12. Proof of Proposition 2

We first consider the one-dimensional case. Clearly it suffices to consider the case that $a_t > 0$. By Theorem 3 and Proposition 5, it follows that

$$(12.1) \qquad \lim_{t\to\infty} P_0^{1;(r,0)}(\tau_{a_t} > t) = \begin{cases} 0, \text{ if } \lim_{t\to\infty} t\lambda_0(r,0;a_t) = \infty; \\ 1, \text{ if } \lim_{t\to\infty} t\lambda_0(r,0;a_t) = 0. \end{cases}$$

Using (1.9), we can replace $\lambda_0(r,0;a_t)$ in (12.1) by $re^{-a_t\sqrt{\frac{2(r-\lambda_0(r,0;a_t))}{D}}}$. Thus,

$$(12.2)$$
$$\lim_{t\to\infty} P_0^{1;(r,0)}(\tau_{a_t} > t) = \begin{cases} 0, \text{ if } \lim_{t\to\infty} \left(a_t - \sqrt{\frac{D}{2(r-\lambda_0(r,0;a_t))}} \log t\right) = -\infty; \\ 1, \text{ if } \lim_{t\to\infty} \left(a_t - \sqrt{\frac{D}{2(r-\lambda_0(r,0,a_t))}} \log t\right) = \infty. \end{cases}$$

The case $\lim_{t\to\infty}(a_t - \sqrt{\frac{D}{2r}} \log t) = -\infty$ in (1.4) follows from (12.2).

We now consider (1.4) in the case that

$$\text{(12.3)} \qquad \lim_{t\to\infty} \left(a_t - \sqrt{\frac{D}{2r}}\log t\right) = \infty.$$

By Corollary 1, $\lambda_0(r,0;a) \le re^{-c(r,a,D)\sqrt{\frac{2r}{D}}a}$, where $\lim_{a\to\infty} c(r,a,D) = 1$. Thus,

$$\sqrt{\frac{D}{2(r-\lambda_0(r,0;a_t))}} \le \sqrt{\frac{D}{2r}}\left(1 - e^{-c(r,a_t,D)\sqrt{\frac{2r}{D}}a_t}\right)^{-\frac{1}{2}}.$$

Since (12.3) holds, we have $e^{-c(r,a_t,D)\sqrt{\frac{2r}{D}}a_t} \le t^{-\frac{1}{2}}$, for all large t. Consequently,

$$\text{(12.4)} \qquad \sqrt{\frac{D}{2(r-\lambda_0(r,0;a_t))}} \le \sqrt{\frac{D}{2r}}(1 + t^{-\frac{1}{2}}), \text{ for large } t.$$

From (12.4), we conclude that if (12.3) holds, then

$$\lim_{t\to\infty}\left(a_t - \sqrt{\frac{D}{2(r-\lambda_0(r,0;a_t))}}\log t\right) = \infty,$$

and consequently, from (12.2), $\lim_{t\to\infty} P_0^{1;(r,0)}(\tau_{a_t} > t) = 1$. This concludes the proof of (1.4) in the case that (12.3) holds.

We now turn to the multi-dimensional case. By Theorem 3-Bessel, Proposition 4-Bessel, (1.19) and (1.20), it follows that

$$\text{(12.5)} \qquad \lim_{t\to\infty} P_0^{d;(r,0)}(\tau_{a_t} > t) = \begin{cases} 0, & \text{if } \lim_{t\to\infty} t\lambda_0(r,|a_t|;\epsilon_0) = \infty; \\ 1, & \text{if } \lim_{t\to\infty} t\lambda_0(r,|a_t|;\epsilon_0) = 0. \end{cases}$$

By (10.26), we can replace $\lambda_0(r,|a_t|;\epsilon_0)$ by $|a_t|^{\frac{1-d}{2}}e^{-\sqrt{(r-\lambda_0(r,|a_t|;\epsilon_0))\frac{2}{D}}|a_t|}$ in (12.5). Thus,

(12.6)
$$\lim_{t\to\infty} P_0^{d;(r,0)}(\tau_{|a_t|} > t) = 0, \text{ if}$$

$$\lim_{t\to\infty}\left(|a_t| - \sqrt{\frac{D}{2(r-\lambda_0(r,|a_t|;\epsilon_0))}}\log t + \frac{d-1}{2}\sqrt{\frac{D}{2(r-\lambda_0(r,|a_t|;\epsilon_0))}}\log|a_t|\right) = -\infty;$$

(12.7)
$$\lim_{t\to\infty} P_0^{d;(r,0)}(\tau_{a_t} > t) = 1, \text{ if}$$

$$\lim_{t\to\infty}\left(|a_t| - \sqrt{\frac{D}{2(r-\lambda_0(r,|a_t|;\epsilon_0))}}\log t + \frac{d-1}{2}\sqrt{\frac{D}{2(r-\lambda_0(r,|a_t|;\epsilon_0))}}\log|a_t|\right) = \infty.$$

We first consider the first case in (1.5) and thus assume that

$$(12.8) \quad \lim_{t\to\infty}(|a_t| - \sqrt{\frac{D}{2r}}\log t + \gamma \log\log t) = -\infty, \text{ for some } \gamma > \frac{d-1}{2}\sqrt{\frac{D}{2r}}.$$

Without loss of generality we may assume that $\lim_{t\to\infty}|a_t| = \infty$, since otherwise $\lim_{t\to\infty}P_0^{d;(r,0)}(\tau_{a_t} > t) = 0$ follows trivially. Then we have $\log|a_t| \le \log\log t + C$, for some constant C, and $\lim_{t\to\infty}\lambda_0(r, |a_t|; \epsilon_0) = 0$. From this and (12.8), it follows that the condition in the second line of (12.6) holds, and consequently, $\lim_{t\to\infty}P_0^{d;(r,0)}(\tau_{a_t} > t) = 0$.

Now we consider the second case in (1.5) and thus assume that

$$(12.9) \quad \lim_{t\to\infty}(|a_t| - \sqrt{\frac{D}{2r}}\log t + \frac{d-1}{2}\sqrt{\frac{D}{2r}}\log\log t) = \infty.$$

By (10.26), we have for sufficiently large A, $\lambda_0(r, A; \epsilon_0) \le re^{-\sqrt{(r-\lambda_0(r,A;\epsilon_0))\frac{2}{D}}A}$, where we have included the r for convenience in the next step. Thus, for sufficiently large t,

$$(12.10) \quad \begin{aligned} \sqrt{\frac{D}{2(r-\lambda_0(r,|a_t|;\epsilon_0))}} &\le \sqrt{\frac{D}{2r}}(1 - e^{-\sqrt{\frac{r}{D}}|a_t|})^{-\frac{1}{2}} \le \\ \sqrt{\frac{D}{2r}}(1 - e^{-\sqrt{\frac{r}{D}}\sqrt{\frac{D}{4r}}\log t})^{-\frac{1}{2}} &= \sqrt{\frac{D}{2r}}(1 - t^{-\frac{1}{2}})^{-\frac{1}{2}} \le \sqrt{\frac{D}{2r}}(1 + t^{-\frac{1}{2}}). \end{aligned}$$

From (12.10) and (12.9), it follows that the second line in (12.7) holds, and consequently, $\lim_{t\to\infty}P_0^{d;(r,0)}(\tau_{B(A_t)} > t) = 1$. $\qquad\square$

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