CLUSTERING OF CONSECUTIVE NUMBERS IN PERMUTATIONS AVOIDING A PATTERN OF LENGTH THREE OR AVOIDING A FINITE NUMBER OF SIMPLE PATTERNS

ROSS G. PINSKY

Abstract. For $\eta \in S_3$, let $S_n^{\text{av}(\eta)}$ denote the set of permutations in S_n that avoid the pattern η, and let $E_n^{\text{av}(\eta)}$ denote the expectation with respect to the uniform probability measure on $S_n^{\text{av}(\eta)}$. For $n \ge k \ge 2$ and $\tau \in S_k^{\text{av}(\eta)}$, let $N_n^{(k)}(\sigma)$ denote the number of occurrences of k consecutive numbers appearing in k consecutive positions in $\sigma \in S_n^{\text{av}(\eta)}$, and let $N_n^{(k;\tau)}(\sigma)$ denote the number of such occurrences for which the order of the appearance of the k numbers is the pattern τ. We obtain explicit formula formulas for $E_n^{\text{av}(\eta)} N_n^{(k;\tau)}$ and $E_n^{\text{av}(\eta)} N_n^{(k)}$, for all $2 \le k \le n$, all $\eta \in S_3$ and all $\tau \in S_k^{\text{av}(\eta)}$. These exact formulas then yield asymptotic formulas as $n \to \infty$ with k fixed, and as $n \to \infty$ with $k = k_n \to \infty$. We also obtain analogous results for $S_n^{\text{av}(\eta_1,\cdots,\eta_r)}$, the subset of S_n consisting of permutations avoiding the patterns $\{\tau_i\}_{i=1}^r$, where $\tau_i \in S_{m_i}$, in the case that $\{\tau_i\}_{i=1}^n$ are all simple permutations. A particular case of this is the set of separable permutations, which corresponds to $r = 2$, $\tau_1 = 2413, \tau_2 = 3142$.

1. Introduction and Statement of Results

Let $k, n \in \mathbb{N}$ with $2 \le k \le n$. Let P_n denote the uniform probability measure on the set S_n of permutations of $[n] := \{1, \cdots, n\}$ and denote by E_n expectations corresponding to P_n. Denote a permutation $\sigma \in S_n$ by $\sigma = \sigma_1\sigma_2\cdots\sigma_n$. The set of k consecutive numbers $\{l, l+1, \cdots, l+k-1\} \subset [n]$ appears in a set of consecutive positions in the permutation if there exists

2010 *Mathematics Subject Classification.* 60C05,05A05.

Key words and phrases. random permutation; pattern avoiding permutation; clustering, simple pattern, separable permutation .

an m such that $\{l, l+1, \cdots, l+k-1\} = \{\sigma_m, \sigma_{m+1}, \cdots, \sigma_{m+k-1}\}$. Let $A_n^{k;l} \subset S_n$ denote the event that the set of k consecutive numbers $\{l, l+1, \cdots, l+k-1\}$ appears in a set of consecutive positions. It is immediate that for any $1 \le l, m \le n-k+1$, the probability that $\{l, l+1, \cdots, l+k-1\} = \{\sigma_m, \sigma_{m+1}, \cdots, \sigma_{m+k-1}\}$ is equal to $\frac{k!(n-k)!}{n!}$. Thus,

$$P_n(A_n^{k;l}) = (n-k+1)\frac{k!(n-k)!}{n!} \sim \frac{k!}{n^{k-1}}, \text{ as } n \to \infty, \text{for } k \ge 2. \tag{1.1}$$

Let $N_n^{(k)} = \sum_{l=1}^{n-k+1} 1_{A_n^{k;l}}$ denote the number of sets of k consecutive numbers appearing in sets of consecutive positions, and let $A_n^k = \cup_{l=1}^{n-k+1} A_n^{k;l}$ denote the event that there exists a set of k consecutive numbers appearing in a set of consecutive positions. Then

$$E_n N_n^{(k)} = (n-k+1)^2 \frac{k!(n-k)!}{n!} \sim \frac{k!}{n^{k-2}}, \text{ as } n \to \infty, \text{ for } k \ge 2. \tag{1.2}$$

Using inclusion-exclusion along with (1.1), it is not hard to show that

$$P_n(A_n^k) \sim \frac{k!}{n^{k-2}}, \text{ as } n \to \infty, \text{ for } k \ge 3. \tag{1.3}$$

It follows from (1.2) (or from (1.3)) that for $k \ge 3$, the sequence $\{N_n^{(k)}\}_{n=1}^\infty$ converges to zero in probability. On the other hand, $\{N_n^{(2)}\}_{n=1}^\infty$ converges in distribution to a Poisson random variable with parameter 2. This result on the clustering of consecutive numbers in permutations goes back over 75 years; see [12], [6].

In this article, we study the clustering of consecutive numbers in permutations avoiding a pattern of length three, as well as in permutations that avoid a fixed number of patterns, all of which are *simple,* the definition of which is given below. We recall the definition of pattern avoidance for permutations. If $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ and $\eta = \eta_1\cdots\eta_m \in S_m$, where $2 \le m \le n$, then we say that σ contains η as a pattern if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_m \le n$ such that for all $1 \le j, k \le m$, the inequality $\sigma_{i_j} < \sigma_{i_k}$ holds if and only if the inequality $\eta_j < \eta_k$ holds. If σ does not contain η, then we say that σ *avoids* η. We denote by $S_n^{\text{av}(\eta)}$ the set of permutations in S_n that avoid η. If $n < m$, we define $S_n^{\text{av}(\eta)} = S_n$. We denote by $P_n^{\text{av}(\eta)}$ the uniform probability measure on $S_n^{\text{av}(\eta)}$, and denote by $E_n^{\text{av}(\eta)}$ expectations corresponding to $P_n^{\text{av}(\eta)}$.

The main results in this paper concern permutations avoiding a pattern of length three. Let $\eta \in S_3$, $k \geq 2$ and $\tau \in S_k^{\text{av}(\eta)}$. For $\sigma \in S_n^{\text{av}(\eta)}$, with $n \geq k$, let $N_n^{(k;\tau)}(\sigma)$ denote the number of occurrences of k consecutive numbers appearing in k consecutive positions in σ, and such that the order of their appearance is the pattern τ. (Such an occurrence is defined by the existence of $1 \leq l, m \leq n-k+1$ such that $\{l, l+1, \cdots, l+k-1\} = \{\sigma_m, \cdots, \sigma_{m+k-1}\}$ and in addition, $\tau_i = \sigma_{m+i-1} - (l-1)$, $i = 1, \cdots, k$.) Let $N_n^{(k)}(\sigma)$ denote the number of occurrences of k consecutive numbers appearing in k consecutive positions in σ, without regard to the order of their appearance; that is, $N_n^{(k)}(\sigma) = \sum_{\tau \in S_k^{\text{av}(\eta)}} N_n^{(k;\tau)}(\sigma)$. We obtain explicit formulas for $E_n^{\text{av}(\eta)} N_n^{(k;\tau)}$ and $E_n^{\text{av}(\eta)} N_n^{(k)}$, for all $2 \leq k \leq n$, all $\eta \in S_3$ and all $\tau \in S_k^{\text{av}(\eta)}$. (Of course, $P_n^{\text{av}(\eta)}(N_n^{(k;\tau)} = 0) = 1$, if $\tau \in S_k - S_k^{\text{av}(\eta)}$.) These exact formulas then yield asymptotic formulas as $n \to \infty$ with k fixed, and as $n \to \infty$ with $k = k_n \to \infty$.

Although there are six permutations η in S_3, it suffices to consider just two of them—one from $\{132, 213, 231, 312\}$ and one from $\{123, 321\}$. Indeed, recall that the *reverse* of a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is the permutation $\sigma^{\text{rev}} := \sigma_n \cdots \sigma_1$, and the *complement* of σ is the permutation σ^{com} satisfying $\sigma_i^{\text{com}} = n+1-\sigma_i$, $i = 1, \cdots, n$. Let $\sigma^{\text{rev-com}}$ denote the permutation obtained by applying reversal and then complementation to σ (or equivalently, vice versa). It is immediate that the quantity $E_n^{\text{av}(\eta)} N_n^{(k;\tau)}$ remains unchanged if each of η and τ is replaced by its reversal, or by its complementation, or by its complementation-reversal. Thus, it suffices to consider, say 231 and 321, since the three permutations 132, 213 and 312 are obtained from 231 respectively by reversal, complementation and complementaion-reversal, and the permutation 123 is obtained from 321 by reversal. We will prove our results for the patterns 231 and 321, but we state them in complete generality.

Denote the nth Catalan number by C_n: $C_n = \frac{1}{n+1}\binom{2n}{n}$. As is well known, $|S_n^{\text{av}(\eta)}| = C_n$, for all $\eta \in S_3$ and all $n \in \mathbb{N}$ [3, 11].

Theorem 1. *Let $\eta \in \{132, 213, 231, 312\}$.*
i. Let $2 \le k \le n$ and define $\tau^*(\eta) \in S_k^{av(\eta)}$ by

$$\tau^*(\eta) = \begin{cases} k \cdots 1, & \text{if } \eta = 231 \text{ or } 312; \\ 1 \cdots k, & \text{if } \eta = 132 \text{ or } \eta = 213. \end{cases} \tag{1.4}$$

For $\tau \in S_k^{av(\eta)}$,

$$E_n^{av(\eta)} N_n^{(k;\tau)} = \begin{cases} \frac{(n-k+2)C_{n-k+1}}{2C_n}, \text{if } \tau \neq \tau^*(\eta); \\ \frac{(n-k+3)C_{n-k+2}}{2C_n} - \frac{(n-k+2)C_{n-k+1}}{C_n}, \text{if } \tau = \tau^*(\eta). \end{cases} \tag{1.5}$$

Also,

$$\lim_{n\to\infty} \frac{N_n^{(k;\tau)}}{E_n^{av(\eta)} N_n^{(k;\tau)}} = 1 \text{ in probability.} \tag{1.6}$$

ii.

$$E_n^{av(\eta)} N_n^{(k)} = \frac{C_{n-k+1}}{2C_n}\Big((n-k+2)C_k + n - k\Big). \tag{1.7}$$

Also,

$$\lim_{n\to\infty} \frac{N_n^{(k)}}{E_n^{\text{av}(\eta)} N_n^{(k)}} = 1 \text{ in probability.} \tag{1.8}$$

Remark. It is easy to check that $E_n^{\text{av}(\eta)} N_n^{(k;\tau)}$ is larger for $\tau = \tau^*(\eta)$ than for $\tau \neq \tau^*(\eta)$, for all $n > k$. (For $n = k$, they are both of course equal to $\frac{1}{C_k}$.)

Using the fact that $C_n \sim \frac{4^n}{\sqrt{\pi} n^{\frac{3}{2}}}$, the following corollary of Theorem 1 follows by straightforward calculation.

Corollary 1. *Let $\eta \in \{132, 213, 231, 312\}$ and let $\tau^*(\eta)$ be as in (1.4).*
i. Let $\tau^*(\eta) \neq \tau \in S_k^{av(\eta)}$. Then

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} E_n^{av(\eta)} N_n^{(k;\tau)} = \frac{1}{2 \cdot 4^{k-1}}; \\ &E_n^{av(\eta)} N_n^{(k_n;\tau)} \sim \frac{n}{2 \cdot 4^{k_n - 1}}, \text{ if } k_n = o(n). \end{aligned} \tag{1.9}$$

ii. Let $\tau = \tau^*(\eta)$*. Then*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} E_n^{av(\eta)} N_n^{(k;\tau)} = \frac{1}{4^{k-1}}; \\ &E_n^{av(\eta)} N_n^{(k_n;\tau)} \sim \frac{n}{4^{k_n-1}}, \text{ if } k_n = o(n). \end{aligned} \tag{1.10}$$

iii.

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} E_n^{av(\eta)} N_n^{(k)} = \frac{C_k + 1}{2 \cdot 4^{k-1}}; \\ &E_n^{av(\eta)} N_n^{(k_n)} \sim \frac{2}{\sqrt{\pi}} \frac{n}{k_n^{\frac{3}{2}}}, \text{ if } \lim_{n\to\infty} k_n = \infty \text{ and } k_n = o(n). \end{aligned} \tag{1.11}$$

Remark. From (1.11), it follows that $E_n^{\mathrm{av}(\eta)} N_n^{(k_n)}$ remains bounded away from zero when $n \to \infty$ if and only if $k_n = O(n^{\frac{2}{3}})$, and from (1.9) and (1.10) it follows that $E_n^{\mathrm{av}(\eta)} N_n^{(k_n;\tau)}$ remains bounded away from zero when $n \to \infty$ if and only if $\limsup_{n\to\infty}(k_n - \frac{\log n}{\log 4}) < \infty$.

Theorem 2. *Let* $\eta \in \{123, 321\}$*.*
i. Let $2 \le k \le n$ *and define* $\tau^*(\eta) \in S_k^{av(\eta)}$ *by*

$$\tau^*(\eta) = \begin{cases} k \cdots 1, & \text{if } \eta = 123; \\ 1 \cdots k, & \text{if } \eta = 321. \end{cases} \tag{1.12}$$

For $\tau \in S_k^{av(\eta)}$*,*

$$E_n^{av(\eta)} N_n^{(k;\tau)} = \begin{cases} \frac{C_{n-k+1}}{C_n}, & \text{if } \tau \ne \tau^*(\eta); \\ (n-k+1)\frac{C_{n-k+1}}{C_n}, & \text{if } \tau = \tau^*(\eta). \end{cases} \tag{1.13}$$

ii.

$$E_n^{av(\eta)} N_n^{(k)} = \frac{C_{n-k+1}(n-k+C_k)}{C_n}. \tag{1.14}$$

Remark 1. From (1.5) and (1.13), it follows that for each $k \ge 2$, $E_n^{\mathrm{av}(\eta)} N_n^{(k;\tau)}$ has linear growth in n, for all pairs (η, τ), with $\eta \in \{132, 213, 231, 312\}$ and $\tau \in S_k^{\mathrm{av}(\eta)}$, and for the pairs $\eta = 123, \tau = k \cdots 1$ and $\eta = 321, \tau = 1 \cdots k$. On the other hand $E_n^{\mathrm{av}(\eta)} N_n^{(k;\tau)}$ is bounded and bounded away from zero for $\eta = 123$ and $k \cdots 1 \ne \tau \in S_k^{\mathrm{av}(123)}$ and for $\eta = 321$ and $1 \cdots k \ne \tau \in S_k^{\mathrm{av}(321)}$.

Remark 2. We elaborate on the behavior in the case of the pairs $\eta = 123$ and $kk-1\cdots 1 \neq \tau \in S_k^{\text{av}(123)}$, and the pairs $\eta = 321$ and $12\cdots k \neq \tau \in S_k^{\text{av}(321)}$. Consider, for example, $\eta = 321$ and $12\cdots k \neq \tau \in S_k^{\text{av}(321)}$. As is well known, every permutation in $S_n^{\text{av}(321)}$ is composed of two increasing subsequences. (The two subsequences are not necessarily unique; for example, the permutation $145236798 \in S_9^{\text{av}(321)}$ is composed of the two increasing subsequence 12379 and 4568 as well as of the two increasing subsequences 12368 and 4579.) A cluster of length k and pattern τ in $\sigma \in S_n^{\text{av}(321)}$ will necessarily have to include numbers from both increasing subsequences corresponding to σ. Theorem 2 states that the expected number of such sequences is bounded (and bounded away from zero) as $n \to \infty$. This means that the two subsequences have very little intertwining.

The following corollary of Theorem 2 follows by straightforward calculation.

Corollary 2. *Let $\eta \in \{123, 321\}$ and let $\tau^*(\eta)$ be as in* (1.12).
i. Let $\tau^*(\eta) \neq \tau \in S_k^{av(\eta)}$. Then

$$\lim_{n\to\infty} E_n^{av(\eta)} N_n^{(k;\tau)} = \frac{1}{4^{k-1}};$$
$$E_n^{av(\eta)} N_n^{(k_n;\tau)} \sim \frac{1}{4^{k_n-1}}, \ \textit{if } k_n = o(n).$$

ii. Let $\tau = \tau^*(\eta)$. Then

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} E_n^{av(\eta)} N_n^{(k;\tau)} = \frac{1}{4^{k-1}}; \\ &E_n^{av(\eta)} N_n^{(k_n;\tau)} \sim \frac{n}{4^{k_n-1}}, \ \textit{if } k_n = o(n). \end{aligned} \tag{1.15}$$

iii.

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} E_n^{av(\eta)} N_n^{(k)} = \frac{1}{4^{k-1}}; \\ &E_n^{av(\eta)} N_n^{(k)} \sim \frac{n}{4^{k_n-1}} + \frac{4}{\sqrt{\pi} k_n^{\frac{3}{2}}}, \ \textit{if } \lim_{n\to\infty} k_n = \infty \textit{ and } k_n = o(n). \end{aligned} \tag{1.16}$$

Remark. From (1.15) and (1.16), it follows that each of $E_n^{\text{av}(\eta)} N_n^{(k_n)}$ and $E_n^{\text{av}(\eta)} N_n^{(k_n;\tau^*(\eta))}$ remains bounded away from zero when $n \to \infty$ if and only if $\limsup_{n\to\infty}(k_n - \frac{\log n}{\log 4}) < \infty$.

The methods of proof for Theorem 1 and Theorem 2 are completely different from one another. The proof of Theorem 1 exploits generating functions, whereas the proof of Theorem 2 is much more of a purely combinatorial argument. The method of proof of Theorem 2 easily extends to allow one to obtain similar results for permutations avoiding simple patterns. A permutation $\eta \in S_m$ is called *simple* if $\eta \notin A^{(m)}_{k;l}$, for all $k \in \{2, \cdots, m-1\}$ and all $l \in \mathbb{N}$ satisfying $k + l - 1 \le m$. (Equivalently, η is simple if and only if $\{\eta_a, \cdots, \eta_{a+k-1}\}$ is not equal to a block of k consecutive numbers in $[m]$, for all $k \in \{2, \cdots, m-1\}$ and all $a \in \mathbb{N}$ satisfying $a+k-1 \le m$.) For example, 6241753 is a simple permutation in S_7, but 6241375 is not, because of the block 2413. Note that there are no simple permutations in S_2 or S_3.

For $r \in \mathbb{N}$ and a collection of permutations $\{\eta_i\}_{i=1}^r$, with $\eta_i \in S_{m_i}$, where $m_i \ge 2$, denote by $S_n^{\text{av}(\eta_1,\cdots,\eta_r)}$ the set of permutations in S_n that avoid all of the patterns $\{\eta_i\}_{i=1}^r$. Denote by $P_n^{\text{av}(\eta_1,\cdots,\eta_r)}$ the uniform probability measure on $S_n^{\text{av}(\eta_1,\cdots,\eta_r)}$, and denote by $E_n^{\text{av}(\eta_1,\cdots,\eta_r)}$ expectations corresponding to $P_n^{\text{av}(\eta_1,\cdots,\eta_r)}$. For $\sigma \in S_n^{\text{av}(\eta_1,\cdots,\eta_r)}$ and $\tau \in S_k^{\text{av}(\eta_1,\cdots,\eta_r)}$, with $2 \le k \le n$, let $N_n^{(k;\tau)}(\sigma)$ denote the number of occurrences of k consecutive numbers appearing in k consecutive positions in σ, and such that the order of their appearance is the pattern τ. Let $N_n^{(k)}(\sigma)$ denote the number of occurrences of k consecutive numbers appearing in k consecutive positions in σ, without regard to the order of their appearance.

Theorem 3. *Let $r \in \mathbb{N}$ and let $\{\eta_i\}_{i=1}^r$, with $\eta_i \in S_{m_i}$ and $m_i \ge 4$, be simple permutations. Then for $2 \le k \le n$ and $\tau \in S_k^{av(\eta_1,\cdots,\eta_m)}$,*

$$E_n^{av(\eta_1,\cdots,\eta_r)} N_n^{(k;\tau)} = \frac{(n-k+1)|S_{n-k+1}^{av(\eta_1,\cdots,\eta_r)}|}{|S_n^{av(\eta_1,\cdots,\eta_m)}|}. \tag{1.17}$$

Also,

$$E_n^{av(\eta_1,\cdots,\eta_m)} N_n^{(k)} = \frac{(n-k+1)|S_{n-k+1}^{av(\eta_1,\cdots,\eta_r)}|\,|S_k^{av(\eta_1,\cdots,\eta_m)}|}{|S_n^{av(\eta_1,\cdots,\eta_m)}|}. \tag{1.18}$$

We elaborate on two particular cases of Theorem 3.

A *separable* permutation is a permutation that can be constructed from the singleton in S_1 via a series of iterations of *direct sums* and *skew sums.*

(See [9], for example, for more details.) An equivalent definition of a separable permutation [4] is a permutation that avoids the two patterns 2413 and 3142. The generating function for the enumeration of separable permutations is known explicitly and allows one to show [5, p. 474-475] that

$$|S_n^{\mathrm{sep}}| \sim \frac{1}{2\sqrt{\pi n^3}}(3-2\sqrt{2})^{-n+\frac{1}{2}}. \tag{1.19}$$

(The sequence on the right hand side of (1.19) is the sequence of big Schröder numbers–A006318 in the *On-Line Encyclopedia of Integer Sequences.)* Let S_n^{sep} denote the set of separable permutations in S_n, let P_n^{sep} denote the uniform probability measure on S_n^{sep} and let E_n^{sep} denote the corresponding expectation. Since the permutations 2413 and 3142 are simple, the following result follows immediately from Theorem 3 and (1.19).

Corollary 3. *For* $2 \le k \le n$ *and* $\tau \in S_k^{sep}$,

$$E_n^{sep} N_n^{(k;\tau)} = \frac{(n-k+1)|S_{n-k+1}^{sep}|}{|S_n^{sep}|},$$

and

$$\lim_{n\to\infty} \frac{E_n^{sep} N_n^{(k;\tau)}}{n} = (3-2\sqrt{2})^{k-1}.$$

Also,

$$E_n^{sep} N_n^{(k)} = \frac{(n-k+1)|S_{n-k+1}^{sep}|\,|S_k^{sep}|}{|S_n^{sep}|},$$

and

$$\lim_{n\to\infty} \frac{E_n^{sep} N_n^{(k)}}{n} = (3-2\sqrt{2})^{k-1}|S_k^{sep}|.$$

One formulation of the Stanley-Wilf conjecture, completely proved in [7], states that for every permutation $\tau \in S_m$, $m \ge 2$, there exists a number $L(\tau) > 1$ such that

$$\lim_{n\to\infty} |S_n(\tau)|^{\frac{1}{n}} = L(\tau).$$

We refer to $L(\tau)$ as the *Stanley-Wilf limit.* Furthermore, it is known that

$$\lim_{n\to\infty} \frac{|S_{n+1}(\tau)|}{|S_n(\tau)|} = L(\tau), \text{ for any simple permutation } \tau \in S_k,\ k \ge 2. \tag{1.20}$$

Indeed, (1.20) was proven in [2] for a wide class of permutations τ, including all simple ones. Thus, the following asymptotic result follows as an immediate corollary of Theorem 3 and (1.20).

Corollary 4. *Let $\eta \in S_m$, $m \geq 4$, be a simple permutation. Then for $2 \leq k \leq n$ and $\tau \in S_k^{av(\eta)}$,*

$$\lim_{n\to\infty} \frac{1}{n} E_n^{av(\eta)} N_n^{(k;\tau)} = \frac{1}{L(\tau)^{k-1}},$$

and

$$\lim_{n\to\infty} \frac{1}{n} E_n^{av(\eta)} N_n^{(k)} = \frac{|S_k^{av(\eta)}|}{L(\tau)^{k-1}},$$

where $L(\tau)$ is the Stanley-Wilf limit.

This leads to a natural question.

Open Question. Is it true that for every $\eta \in \cup_{j=1}^{\infty} S_j$ and all $k \geq 2$, the expectation $E_n^{\text{av}(\eta)} N_n^{(k)}$ grows linearly in n, or equivalently, that there exists a $\tau \in S_k^{\text{av}(\eta)}$ such that the expectation $E_n^{\text{av}(\eta)} N_n^{(k;\tau)}$ grows linearly in n?

The recent paper [10] studied the clustering of consecutive numbers under Mallows distributions.

In section 2, we prove Theorem 1, except for the law of large numbers in (1.6) and (1.8). Of course, (1.8) follows immediately from (1.6). Using the second moment method, (1.6) follows immediately from Corollary 1 and the following proposition, which we prove in section 3.

Proposition 1. *Under the assumptions in Theorem 1, for all $\tau \in S_k^{av(\eta)}$, $Var(N_n^{(k;\tau)}) = o(n^2)$.*

We prove Theorem 2 in section 4 and Theorem 3 in section 5.

2. Proof of Theorem 1

As noted before the statement of the theorem, it suffices to consider the case $\eta = 231$. Let $k \geq 2$ and $\tau \in S_k^{\text{av}(231)}$. We consider $n \geq k$. The following definition will be useful. Let $a_1 < a_2 \cdots < a_m$ be real numbers and let $\rho = \rho_1 \cdots \rho_m$ be a permutation of these numbers. We define $\text{red}(\rho) \in S_m$, the reduction of ρ, to be the permutation in S_m that has the same pattern as ρ. That is, $\text{red}(\rho)=\sigma$ if σ satisfies $\sigma_i<\sigma_j$ whenever $\rho_i<\rho_j$, $i,j\in[m]$.

Every permutation $\sigma\in S_k^{\text{av}(231)}$ has the property that if $\sigma_j=n$, then the numbers $\{1,\cdots,j-1\}$ appear in the first $j-1$ positions in σ (and then of course, the numbers $\{j,\cdots,n-1\}$ appear in the last $n-j$ positions in σ.) This fact will be used frequently in the proof without further comment. From this fact, along with the fact that $|S_n^{\text{av}(\eta)}|=C_n$, it follows that

$$P_n^{\text{av}(231)}(\sigma_j=n)=\frac{C_{j-1}C_{n-j}}{C_n},\ \text{for } j\in[n]. \tag{2.1}$$

It also follows that under the conditioned measure $P_n^{\text{av}(231)}|\{\sigma_j=n\}$, the permutation $\sigma_1\cdots\sigma_{j-1}\in S_{j-1}$ has the distribution $P_{j-1}^{\text{av}(231)}$, the permutation $\text{red}(\sigma_{j+1}\cdots\sigma_n)$ has the distribution $P_{n-j}^{\text{av}(231)}$, and these two permutations are independent. Note that the following well-known recursion formula for the Catalan numbers follows from (2.1).

$$C_n=\sum_{j=1}^n C_{j-1}C_{n-j},\ n\in\mathbb{N}. \tag{2.2}$$

The key to proving the theorem is the following proposition, whose rather long technical proof will be postponed until we have completed the proof of the theorem.

Proposition 2. *For each $m\in\mathbb{N}$, each $\sigma\in S_m^{av(231)}$ and each $A\subset S_m^{av(231)}$, let the random variables $N_m^{(k;\tau)}(\cdot)$, $1_\sigma(\cdot)$ and $1_A(\cdot)$ be defined on the probability space $(S_m,P_m^{av(231)})$.*
i. Let $k\,k-1\cdots1\neq\tau\in S_k^{av(231)}$. Denote $i_k=\tau_k^{-1}$. Then

$$N_n^{(k;\tau)}|\{\sigma_j=n\}\stackrel{dist}{=}N_{j-1}^{(k;\tau)}+N_{n-j}^{(k;\tau)}+1_{\{j=n-k+i_k\}}1_{A^l_{n-k+i_k-1}}1_{\sigma^{*,k-i_k}}, \tag{2.3}$$

where

$$\sigma^{*,k-i_k}=red(\tau_{i_k+1}\tau_{i_k+2}\cdots\tau_k)\in S_{k-i_k}^{av(231)} \tag{2.4}$$

and

$$A^l_{n-k+i_k-1}=\{\sigma\in S^{av(231)}_{n-k+i_k-1}:\sigma_{n-k+l}=\tau_l+n-k,\ l=1\cdots i_k-1\}, \tag{2.5}$$

and where $N_{j-1}^{(k,\tau)}$ *is independent of* $N_{n-j}^{(k,\tau)}$, *and the pair* $N_{n-k+i_k-1}^{(k,\tau)}, 1_{A^l_{n-k+i_k-1}}$ *is independent of the pair* $N_{k-i_k}^{(k,\tau)}, 1_{\sigma^*,k-i_k}$. *If* $i_k = k$, *then we understand* $1_{\sigma^*,k-i_k}$ *to be the constant 1, and if* $i_k = 1$, *then we understand* $A^l_{n-k+i_k-1}$ *to be the constant 1.*

ii. Let $\tau = k\,k-1\,\cdots 1$. *Then*

$$N_n^{(k;\tau)}|\{\sigma_j = n\} \stackrel{dist}{=} N_{j-1}^{(k;\tau)} + N_{n-j}^{(k;\tau)} + 1_{\{j\le n-k+1\}}1_{A^r_{n-j}}, \tag{2.6}$$

where

$$A^r_{n-j} = \{\sigma \in S_{n-j} : \sigma_l = n-j+1-l,\ l = 1,\cdots,k-1\}, \tag{2.7}$$

and where $N_{j-1}^{(k,\tau)}$ *is independent of the pair* $N_{n-j}^{(k,\tau)}, 1_{A_{n-j}}$.

(The superscripts l and r in (2.5) and (2.7) are just used to distinguish the two sets, and stand for "left" and "right", because of how they arise in the proof.)

Let $s_n^\tau = E_n^{\text{av}(231)} N_n^{(k;\tau)}$. Of course, $s_n^\tau = 0$, for $n = 1,\cdots,k-1$. For convenience, define $s_0^\tau = 0$. The following result follows easily from Proposition 2.

Proposition 3. *For* $2 \le k \le n$,

$$s_n^\tau = \begin{cases} 2\sum_{j=1}^n \frac{C_{j-1}C_{n-j}}{C_n} s_{j-1}^\tau + \frac{C_{n-k}}{C_n}, & \text{if } k\,k-1\,\cdots 1 \ne \tau \in S_k^{av(231)}; \\ 2\sum_{j=1}^n \frac{C_{j-1}C_{n-j}}{C_n} s_{j-1}^\tau + \frac{C_{n-k+1}}{C_n}, & \text{if } \tau = k\,k-1\,\cdots 1. \end{cases} \tag{2.8}$$

Proof. From (2.1), (2.5) and (2.7), it follows that

$$P_n^{\text{av}(231)}(A_n^l) = \frac{C_{n-k}}{C_{n-k+i_k-1}};\quad P_n^{\text{av}(231)}(A_n^r) = \frac{C_{n-j-k+1}}{C_{n-j}}. \tag{2.9}$$

From (2.1), (2.3) and (2.9), we have

$$s_n^\tau = \sum_{j=1}^n \frac{C_{j-1}C_{n-j}}{C_n}(s_{j-1}^\tau + s_{n-j}^\tau) + \frac{C_{n-k+i_k-1}C_{k-i_k}}{C_n}\frac{C_{n-k}}{C_{n-k+i_k-1}}\frac{1}{C_{k-i_k}} =$$

$$2\sum_{j=1}^n \frac{C_{j-1}C_{n-j}}{C_n} s_{j-1}^\tau + \frac{C_{n-k}}{C_n},\ \text{for } k\,k-1\cdots 1 \ne \tau \in S_k^{\text{av}(231)},$$

and from (2.1), (2.6) and (2.9), we have

$$s_n^\tau=\sum_{j=1}^n\frac{C_{j-1}C_{n-j}}{C_n}(s_{j-1}^\tau+s_{n-j}^\tau)+\sum_{j=1}^{n-k+1}\frac{C_{j-1}C_{n-j}}{C_n}\frac{C_{n-j-k+1}}{C_{n-j}}=$$
$$2\sum_{j=1}^n\frac{C_{j-1}C_{n-j}}{C_n}s_{j-1}^\tau+\frac{C_{n-k+1}}{C_n},\ \text{for } \tau=k\,k-1\cdots1,$$

where the last equality follows from (2.2). □

Define

$$G^{(k,\tau)}(t)=\sum_{j=k}^\infty C_js_j^\tau t^j, \tag{2.10}$$

and let

$$C(t)=\sum_{j=0}^\infty C_jt^j$$

denote the generating function of the Catalan numbers. As is well-known [8],

$$C(t)=\frac{1-\sqrt{1-4t}}{2t},\ |t|\le\frac14. \tag{2.11}$$

We use Proposition 3 to prove the following result.

Proposition 4. *For* $|t|\le\frac14$,

$$G^{(k,\tau)}(t)=\begin{cases}\frac{t^kC(t)}{1-2tC(t)}, & \text{if } k\,k-1\cdots1\ne\tau\in S_k^{av(231)};\\ \frac{t^{k-1}(C(t)-1)}{1-2tC(t)}, & \text{if } \tau=k\,k-1\cdots1.\end{cases} \tag{2.12}$$

Proof. Multiplying the first line of (2.8) by C_nt^n and summing over n from k to ∞ gives

$$G^{(k,\tau)}(t)=2tC(t)G^{(k,\tau)}(t)+t^kC(t),\ k\,k-1\cdots1\ne\tau\in S_k,$$

and solving above for $G^{(k,\tau)}(t)$ gives (2.12) for $\tau\ne k\,k-1\cdots1$. Multiplying the second line of (2.8) by C_nt^n and summing over n from k to ∞ gives

$$G^{(k,\tau)}(t)=2tC(t)G^{(k,\tau)}(t)+t^{k-1}(C(t)-1),\ \tau=k\,k-1\cdots1,$$

and solving above for $G^{(k,\tau)}(t)$ gives (2.12) for $\tau=k\,k-1\cdots1$.

□

We now use Proposition 4 to prove Theorem 1.

Proof of Theorem 1. Part (i). From (2.11), we have

$$\frac{tC(t)}{1-2tC(t)} = \frac{1}{2}\big((1-4t)^{-\frac{1}{2}}-1\big). \tag{2.13}$$

A direct calculation [8, p.41] reveals that

$$\frac{\big((1-4t)^{\frac{1}{2}}\big)^{(n)}|_{t=0}}{n!} = -\frac{2}{2n-1}\binom{2n-1}{n},\ n\ge 2. \tag{2.14}$$

From (2.14) and the fact that $\big((1-4t)^{\frac{1}{2}}\big)' = -2(1-4t)^{-\frac{1}{2}}$, it follows that

$$\begin{aligned}&\frac{\big((1-4t)^{-\frac{1}{2}})\big)^{(n)}|_{t=0}}{n!} = -\frac{1}{2}\frac{\big((1-4t)^{\frac{1}{2}}\big)^{(n+1)}|_{t=0}}{n!} = \\ &-\frac{n+1}{2}\frac{\big((1-4t)^{\frac{1}{2}}\big)^{(n+1)}|_{t=0}}{(n+1)!} = -\frac{n+1}{2}\Big(-\frac{2}{2n+1}\binom{2n+1}{n+1}\Big) = \\ &(n+1)C_n,\ n\ge 1.\end{aligned} \tag{2.15}$$

Now (2.13) and (2.15) give

$$\frac{tC(t)}{1-2tC(t)} = \sum_{n=1}^{\infty}\frac{n+1}{2}C_nt^n. \tag{2.16}$$

Consider first the case $k\,k-1\cdots 1 \neq \tau \in S_k$. Then from (2.12) and (2.16)

$$G^{(k,\tau)}(t) = \sum_{n=1}^{\infty}\frac{n+1}{2}C_nt^{n+k-1}, \tag{2.17}$$

which along with (2.10) gives

$$E_n^{\text{av}(231)}N_n^{(k;\tau)} = s_n^{\tau} = \frac{(n-k+2)C_{n-k+1}}{2C_n}, \tag{2.18}$$

which is (1.5) for $\tau \neq k\,k-1\cdots 1$ and $\eta = 231$.

Now consider the case $\tau = k\,k-1\cdots 1$. From (2.11), we have

$$\frac{1}{1-2tC(t)} = (1-4t)^{-\frac{1}{2}}. \tag{2.19}$$

Using this with (2.15) gives

$$\frac{t^{k-1}}{1-2tC(t)} = \sum_{n=0}^{\infty}(n+1)C_nt^{n+k-1}. \tag{2.20}$$

From (2.12), (2.16) and (2.20), we obtain

(2.21)
$$G^{(k,\tau)}(t)=\frac{t^{k-1}C(t)}{1-2tC(t)}-\frac{t^{k-1}}{1-2tC(t)}=\sum_{n=1}^{\infty}\big(\frac{n+2}{2}C_{n+1}-(n+1)C_n\big)t^{n+k-1}.$$

From (2.10) and (2.21), we obtain

$$E_n^{\text{av}(231)}N_n^{(k;\tau)}=s_n^{\tau}=\frac{(n-k+3)C_{n-k+2}}{2C_n}-\frac{(n-k+2)C_{n-k+1}}{C_n}, \tag{2.22}$$

which is (1.5) for $\tau=k\,k-1\cdots 1$ and $\eta=231$. □

Part (ii). From (2.18) and (2.22), we have

$$\begin{aligned} E_n^{\text{av}(231)}N_n^{(k)}=(C_k-1)\frac{(n-k+2)C_{n-k+1}}{2C_n}+\\ \frac{(n-k+3)C_{n-k+2}}{2C_n}-\frac{(n-k+2)C_{n-k+1}}{C_n}.\end{aligned} \tag{2.23}$$

Using the formula $C_{m+1}=\frac{2(2m+1)}{m+2}C_m$ to write C_{n-k+2} in terms of C_{n-k+1} in (2.23), and performing some algebra, gives

$$E_n^{\text{av}(231)}N_n^{(k)}=\frac{C_{n-k+1}}{2C_n}\Big((n-k+2)C_k+n-k\Big),$$

which is (1.7) for $\eta=231$. □

We now return to prove Proposition 2.

Proof of Proposition 2. In order to make the proof more transparent, we first derive it for three particular choices of τ—216345, 621345 and 654321. The proofs of these particular cases will make the general case much easier to follow.

We begin with $\tau=216345$. We show that

(2.24)
$$N_n^{(6;216345)}|\{\sigma_j=n\}\stackrel{\text{dist}}{=}N_{j-1}^{(6;216345)}+N_{n-j}^{(6;216345)}+1_{\{j=n-3\}}1_{A_{n-4}^l}1_{\sigma^{*,3}},$$

where

$$\sigma^{*,3}=123=\text{red}(\tau_4\tau_5\tau_6)\in S_3^{\text{av}(231)} \tag{2.25}$$

and

$$A_{n-4}^l=\{\sigma\in S_{n-4}^{\text{av}(231)}:\sigma_{n-5}=n-4,\sigma_{n-4}=n-5\}, \tag{2.26}$$

and where $N_{j-1}^{(6;216345)}$ is independent of $N_{n-j}^{(6;216345)}$, and the pair $N_{n-4}^{(6;216345)}, 1_{A_{n-4}^l}$ is independent of the pair $N_3^{(6;216345)}, 1_{\sigma^{*,3}}$. The terms $N_{j-1}^{(6;216345)}$ and $N_{n-j}^{(6;216345)}$ on the right hand side of (2.24) are clear; they count the number of clusters of length k and pattern 216345 from $\sigma_1 \cdots, \sigma_{j-1}$ and from $\sigma_{j+1} \cdots \sigma_n$. We now show that the term, $1_{\{j=n-3\}} 1_{A_{n-4}^l} 1_{\sigma^{*,3}}$ counts clusters of length 6 and pattern 216345 that involve the number $n = \sigma_j$. Such a cluster that includes the number $n = \sigma_j$ can only be the cluster

(2.27)

$$\sigma_{j-2} = n-4, \sigma_{j-1} = n-5, \sigma_j = n, \sigma_{j+1} = n-3, \sigma_{j+2} = n-2, \sigma_{j+3} = n-1.$$

Furthermore, since $\sigma \in S_n^{\text{av}(231)}$, (2.27) can only occur if $j = n-3$. Indeed, if $j > n-3$ or $j \leq 2$, then obviously (2.27) cannot occur, while if $3 \leq j < n-3$, then it would follow from (2.27) that $\sigma_{j+4} \leq n-6$. But then $\sigma_{j-1}\sigma_j\sigma_{j+4}$ would have the pattern 231. Finally, given that $\sigma_{n-3} = n$, $\sigma \in S_n^{\text{av}(231)}$ satisfies (2.27) with $j = n-3$ if and only if $\sigma_1 \cdots \sigma_{n-4} \in A_{n-4}^l$ and $\text{red}(\sigma_{n-2}\sigma_{n-1}\sigma_n) = \sigma^{*,3}$ where A_{n-4}^l is as in (2.26) and $\sigma^{*,3}$ is as in (2.25). The above-noted independence follows from the sentence following (2.1). In the sequel, we will refrain from commenting on the justification for independence, as it will always follow from the above-noted sentence.

Consider now the case $\tau = 621345$. We show that

$$N_n^{(6;621345)} | \{\sigma_j = n\} \stackrel{\text{dist}}{=} N_{j-1}^{(6;621345)} + N_{n-j}^{(6;621345)} + 1_{\{j=n-5\}} 1_{\sigma^{*,5}}, \tag{2.28}$$

where

$$\sigma^{*,5} = 21345 = \text{red}(\tau_2 \cdots \tau_6) \tag{2.29}$$

and where $N_{j-1}^{(6;621345)}$ is independent of $N_{n-j}^{(6;621345)}$, and $N_{n-6}^{(6;621345)}$ is independent of the pair $N_5^{(6;621345)}, 1_{\sigma^{*,5}}$. Again, the roles of $N_{j-1}^{(6;621345)}$ and $N_{n-j}^{(6;621345)}$ are clear. We show that the term $1_{\{j=n-5\}} 1_{\sigma^{*,5}}$ counts those clusters of length 6 and pattern 621345 that involve the number $n = \sigma_j$. Such a cluster that includes the number $n = \sigma_j$ can only be the cluster

(2.30)

$$\sigma_j = n, \sigma_{j+1} = n-4, \sigma_{j+2} = n-5, \sigma_{j+3} = n-3, \sigma_{j+4} = n-2, \sigma_{j+5} = n-1.$$

Furthermore, since $\sigma \in S_n^{\text{av}(231)}$, this can occur only if $j = n-5$. Indeed, if $j > n-5$, then obviously (2.30) cannot occur, while if $j < n-5$, then it would follow from (2.30) that $\sigma_{j+6} \le n-6$. But then, for example, $\sigma_{j+1}\sigma_{j+3}\sigma_{j+6}$ would have the pattern 231. Finally, given that $\sigma_{n-5} = n$, $\sigma \in S_n^{\text{av}(231)}$ satisfies (2.30) with $j = n-5$ if and only if $\text{red}(\sigma_{n-4}\cdots\sigma_n) = \sigma^{*,5}$, where $\sigma^{*,5}$ is as in (2.29).

The argument above for $\tau = 621345$ works just as well for any other $\tau \in S_6$ with $\tau_1 = 6$, except for $\tau = 654321$, which we now consider. We will show that

$$N_n^{(6;654321)}|\{\sigma_j = n\} \stackrel{\text{dist}}{=} N_{j-1}^{(6;654321)} + N_{n-j}^{(6;654321)} + 1_{\{j\le n-5\}}1_{A^r_{n-j}}, \tag{2.31}$$

where

$$A^r_{n-j} = \{\sigma \in S_{n-j}^{\text{av}(231)} : \sigma_l = n-j+1-l,\ l = 1,\cdots,5\}, \tag{2.32}$$

and where $N_{j-1}^{(6;654321)}$ is independent of the pair $N_{n-j}^{(6;654321)}, 1_{A^r_{n-j}}$. Again, the roles of $N_{j-1}^{(6;654321)}$ and $N_{n-j}^{(6;654321)}$ are clear. We now show that the term $1_{\{j\le n-5\}}1_{A^r_{n-j}}$ counts clusters of length 6 and pattern 654321 that involve the number $n = \sigma_j$. Such a cluster that includes the number $n = \sigma_j$ can only be the cluster

$$\sigma_j = n, \sigma_{j+1} = n-1, \sigma_{j+2} = n-2, \sigma_{j+3} = n-3, \sigma_{j+4} = n-4, \sigma_{j+5} = n-5. \tag{2.33}$$

Of course, (2.33) cannot occur if $j > n-5$; this accounts for the term $1_{\{j\le n-5\}}$. In the previous case, we also ruled out $j < n-5$ because that would lead to the existence of the pattern 231. In the present case, because the pattern in (2.33) is decreasing, the argument in the above case no longer goes through. Finally, for $j \le n-5$ and given that $\sigma_j = n$, $\sigma \in S_n^{\text{av}(231)}$ satisfies (2.33) if and only if $\text{red}(\sigma_{j+1}\cdots\sigma_n) \in A^r_{n-j}$, where A^r_{n-j} is as in (2.32).

With the above particular cases explained, we now turn to the proof of the general case. Let $i_k = \tau_k^{-1}$. We first assume that $i_k \ne 1$. We will show that part (i) of the proposition holds. The terms $N_{j-1}^{(k,\tau)}$ and $N_{n-j}^{(k;\tau)}$ on the right hand side of (2.3) are clear; they count the number of clusters of length k and pattern τ from $\sigma_1\cdots,\sigma_{j-1}$ and from $\sigma_{j+1}\cdots\sigma_n$. We now show that the term $1_{\{j=n-k+i_k\}}1_{A^l_{n-k+i_k-1}}1_{\sigma^{*,k-i_k}}$ counts clusters of length k and pattern τ that involve the number $n=\sigma_j$. Of course, the only candidate for such a cluster is $\sigma_{j-i_k+1}\sigma_{j-i_k+2}\cdots\sigma_{j+k-i_k}$, and this will indeed constitute such a cluster if and only if

$$\mathrm{red}(\sigma_{j-i_k+1}\sigma_{j-i_k+2}\cdots\sigma_{j+k-i_k})=\tau. \tag{2.34}$$

We now show that (2.34) can only occur if $j=n-k+i_k$. This will account for the term $1_{\{j=n-k+i_k\}}$. Indeed, if $j>n-k+i_k$ or $j\le i_k-1$, then obviously (2.34) cannot occur. Now consider $i_k\le j<n-k+i_k$. Since $\{\sigma_{j-i_k+1},\sigma_{j-i_k+2},\cdots\sigma_{j+k-i_k}\}=\{n-k+1,\cdots,n\}$, it would follow from (2.34) that $\sigma_{j+k-i_k+1}\le n-k$. But then $\sigma_{j-1}\sigma_j\sigma_{j+k-i_k+1}$ would have the pattern 231, which is forbidden. Thus, we conclude that there will be either one such cluster involving n or no such cluster involving n, and the condition for the existence of such a cluster is

$$\mathrm{red}(\sigma_{n-k+1}\cdots\sigma_n)=\tau. \tag{2.35}$$

Finally, given $\sigma_{n-k+i_k}=n$, $\sigma\in S_n^{\mathrm{av}(231)}$ satisfies (2.35) if and only if $\sigma_1\cdots\sigma_{n-k+i_k-1}\in A^l_{n-k+i_k-1}$, where $A^l_{n-k+i_k-1}$ is as in (2.5) and, if $i_k\ne k$, then also $\mathrm{red}(\sigma_{n-k+i_k+1}\cdots\sigma_n)=\sigma^{*,k-i_k}$, where $\sigma^{*,k-i_k}$ is as in (2.4).

We now consider the case that $i_k=1$. Here we need to consider two subcases—the case that $\tau\ne kk-1\cdots1$, and the case that $\tau=kk-1\cdots1$. We first consider the subcase that $\tau\ne kk-1\cdots1$. We will show that part (i) of the proposition holds. Again, the roles of $N^{(k;\tau)}_{j-1}$ and $N^{(k;\tau)}_{n-j}$ are clear. We now show that the term $1_{\{j=n-k+1\}}1_{\sigma^{*,k-1}}$ counts clusters of length k and pattern τ that involve the number $n=\sigma_j$. Of course, the only candidate for such a cluster is $\sigma_j\sigma_{j+1}\cdots\sigma_{j+k-1}$, and this will indeed constitute such a cluster if and only if

$$\mathrm{red}(\sigma_j\sigma_{j+1}\cdots\sigma_{j+k-1})=\tau. \tag{2.36}$$

We now show that (2.36) can only occur if $j = n-k+1$. This will account for the term $1_{\{j=n-k+1\}}$. Indeed, if $j > n-k+1$, then obviously (2.36) cannot occur. Now consider $j < n-k+1$. Since $\{\sigma_j, \sigma_{j+1}, \cdots \sigma_{j+k-1}\} = \{n-k+1, \cdots, n\}$, it would follow from (2.36) that $\sigma_{j+k} \le n-k$. Also, since by assumption, $\tau_2 \cdots \tau_k \ne k-1 \cdots 1$, it follows from (2.36) that there exist indices $l_1 < l_2$ from the set $\{j+1, \cdots, j+k-1\}$ such that $\sigma_{l_1} < \sigma_{l_2}$. But then $\sigma_{l_1}\sigma_{l_2}\sigma_{j+k}$ would have the pattern 231, which is forbidden. Thus, we conclude that there will be either one such cluster involving n or no such cluster involving n, and the condition for the existence of such a cluster is

$$\text{red}(\sigma_{n-k+1} \cdots \sigma_n) = \tau. \tag{2.37}$$

Finally, given $\sigma_{n-k+1} = n$, $\sigma \in S_n^{\text{av}(231)}$ satisfies (2.37) if and only if $\text{red}(\sigma_{n-k+2} \cdots \sigma_n) = \sigma^{*,k-1}$, where $\sigma^{*,k-1}$ is as in (2.4) with $i_k = 1$.

We now turn to the subcase $\tau = k\,k-1 \cdots 1$ of the case $\tau_1 = k$. We will show that part (ii) of the proposition holds. Again, the roles of $N_{j-1}^{(k;\tau)}$ and $N_{n-j}^{(k;\tau)}$ are clear. We now show that the term $1_{\{j \le n-k+1\}} 1_{A^r_{n-j}}$ counts clusters of length k and pattern τ that involve the number $n = \sigma_j$. Of course, the only candidate for such a cluster is $\sigma_j\sigma_{j+1} \cdots \sigma_{j+k-1}$, and this will indeed constitute such a cluster if and only if

$$\text{red}(\sigma_j\sigma_{j+1} \cdots \sigma_{j+k-1}) = \tau = k \cdots 1. \tag{2.38}$$

Of course, (2.38) cannot occur if $j > n-k+1$; this accounts for the term $1_{\{j \le n-k+1\}}$. Finally, given that $\sigma_j = n$, it follows that for $1 \le j \le n-k+1$, $\sigma \in S_n^{\text{av}(231)}$ satisfies (2.38) if and only if $\text{red}(\sigma_{j+1} \cdots \sigma_n) \in A^r_{n-j}$, where A^r_{n-j} is as in (2.7). □

3. Proof of Proposition 1

As noted before the statement of Theorem 1, it suffices to consider the case $\eta = 231$. Let $2 \le k \le n$. We will prove the proposition for the case that $\tau \in S_k^{\text{av}(231)}$ satisfies $\tau \ne k \cdots 1$. The proof uses part (i) of Proposition 2. The case $\tau = k \cdots 1$ is treated similarly, using part (ii) of that proposition.

From part (i) of Proposition 2, it follows that for $2 \le k \le n$,

(3.1)

$$(N_n^{(k;\tau)})^2|\{\sigma_j = n\} \stackrel{\text{dist}}{=} (N_{j-1}^{(k;\tau)})^2 + (N_{n-j}^{(k;\tau)})^2 + 2N_{j-1}^{(k;\tau)}N_{n-j}^{(k;\tau)},\ j \ne n-k+i_k;$$

$$(N_n^{(k;\tau)})^2|\{\sigma_j = n\} \stackrel{\text{dist}}{=} (N_{j-1}^{(k;\tau)})^2 + (N_{n-j}^{(k;\tau)})^2 + 2N_{j-1}^{(k;\tau)}N_{n-j}^{(k;\tau)} +$$

$$2\big(N_{j-1}^{(k;\tau)} + N_{n-j}^{(k;\tau)}\big)1_{A^l_{n-k+i_k-1}}1_{\sigma^{*,k-i_k}} + 1_{A^l_{n-k+i_k-1}}1_{\sigma^{*,k-i_k}},\ j = n-k+i_k,$$

where the notation in (3.1) is as in Proposition 2. Let $r_n^\tau = E_n^{\text{av}(231)}(N_n^{(k;\tau)})^2$ and recall from section 2 that $s_n^\tau = E_n^{\text{av}(231)}N_n^{(k;\tau)}$. From (3.1) and (2.1), along with the independence of certain random variables as indicated in Proposition 2, it follows that

$$r_n^\tau = 2\sum_{j=1}^n \frac{C_{j-1}C_{n-j}}{C_n}r_{j-1}^\tau + 2\sum_{j=1}^n \frac{C_{j-1}C_{n-j}}{C_n}r_{j-1}^\tau s_{j-1}^\tau s_{n-j}^\tau +$$
$$\frac{C_{n-k+i_k-1}C_{k-i_k}}{C_n}\big(2E_{n-k+i_k-1}^{\text{av}(231)}N_{n-k+i_k-1}^{(k;\tau)}1_{A^l_{n-k+i_k-1}}\big)P_{k-i_k}^{\text{av}(231)}(\sigma^{*,k-i_k}) +$$
$$\frac{C_{n-k+i_k-1}C_{k-i_k}}{C_n}P_{n-k+i_k-1}^{\text{av}(231)}(A^l_{n-k+i_k-1})P_{k-i_k}^{\text{av}(231)}(\sigma^{*,k-i_k}),$$

where we have used the fact that $N_n^{k;\tau} = 0$, for $n < k$. Using the fact that $P_{k-i_k}^{\text{av}(231)}(\sigma^{*,k-i_k}) = \frac{1}{C_{k-i_k}}$ along with (2.9), we can rewrite the above equation as

$$\begin{aligned} r_n^\tau &= 2\sum_{j=1}^n \frac{C_{j-1}C_{n-j}}{C_n}r_{j-1}^\tau + 2\sum_{j=1}^n \frac{C_{j-1}C_{n-j}}{C_n}r_{j-1}^\tau s_{j-1}^\tau s_{n-j}^\tau + \\ &2\frac{C_{n-k+i_k-1}}{C_n}E_{n-k+i_k-1}^{\text{av}(231)}N_{n-k+i_k-1}^{(k;\tau)}1_{A^l_{n-k+i_k-1}} + \frac{C_{n-k}}{C_n}. \end{aligned} \tag{3.2}$$

Define

$$W(t) = \sum_{n=k}^\infty C_{n-k+i_k-1}\big(E_{n-k+i_k-1}^{\text{av}(231)}N_{n-k+i_k-1}^{(k;\tau)}1_{A^l_{n-k+i_k-1}}\big)t^n.$$

Note that

(3.3)

$$E_{n-k+i_k-1}^{\text{av}(231)}N_{n-k+i_k-1}^{(k;\tau)}1_{A^l_{n-k+i_k-1}} \le E_{n-k+i_k-1}^{\text{av}(231)}N_{n-k+i_k-1}^{(k;\tau)} = s_{n-k+i_k-1}^\tau.$$

Recalling the definition of $G^{(k;\tau)}(t)$ in (2.10), we note for later use that for each n, the coefficient of t^n in the power series defining $W(t)$ is less than or equal to the coefficient of t^n in the power series for $G^{(k;\tau)}(t)$. This follows from (3.3). Define

$$R^{(k;\tau)}(t)=\sum_{n=k}^{\infty}C_n r_n^{\tau}t^n.$$

Multiplying (3.2) by $C_n t^n$ and summing over n from k to ∞, and recalling the definition of $G^{(k;\tau)}(t)$, we obtain

$$R^{(k;\tau)}(t)=2C(t)R^{(k;\tau)}(t)+2t(G^{(k;\tau)}(t))^2+t^{k-i_k+1}W(t)+t^kC(t),$$

from which it follows that

$$R^{(k;\tau)}(t)=\frac{2t(G^{(k;\tau)}(t))^2+t^{k-i_k+1}W(t)+t^kC(t)}{1-2tC(t)}. \tag{3.4}$$

Using (2.11) and (2.12), we obtain after some algebra,

$$\frac{2t(G^{(k;\tau)}(t))^2}{1-2tC(t)}=(t^{2k-1}-2t^{2k})(1-4t)^{-\frac{3}{2}}-t^{2k-1}(1-4t)^{-1}. \tag{3.5}$$

Also,

$$\frac{t^kC(t)}{1-2tC(t)}=\frac{1}{2}t^{k-1}(1-4t)^{-\frac{1}{2}}-\frac{1}{2}t^{k-1}. \tag{3.6}$$

From (2.15), we have

$$(1-4t)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}(n+1)C_nt^n. \tag{3.7}$$

Differentiating (3.7), we obtain

$$(1-4t)^{-\frac{3}{2}}=\frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)C_{n+1}t^n. \tag{3.8}$$

Since $\frac{t^{k-i_k+1}W(t)}{1-2tC(t)}=t^{k-i_k+1}W(t)(1-4t)^{-\frac{1}{2}}$, and since the coefficients of the power series for $W(t)$ are all nonnegative and are dominated by those of the power series for $G^{(k;\tau)}(t)$, it follows (3.7) that

(3.9)

the coefficients of the power series for $\dfrac{t^{k-i_k+1}W(t)}{1-2tC(t)}$ are dominated by those for the power series $\dfrac{t^{k-i_k+1}G^{(k;\tau)}(t)}{1-2tC(t)}$.

By (2.11) and (2.12),

$$\frac{t^{k-i_k+1}G^{(k;\tau)}(t)}{1-2tC(t)} = \frac{1}{2}t^{2k-i_k}\big((1-4t)^{-1}-(1-4t)^{-\frac{1}{2}}\big). \tag{3.10}$$

Since $C_n \sim \frac{4^n}{\sqrt{\pi}n^{\frac{3}{2}}}$, it follows from (3.4)-(3.10) that the leading order term as $n\to\infty$ of the coefficient $C_n r_n^\tau$ of t^n in the power series for $R^{(k;\tau)}(t)$ comes from the coefficient of t^n in the power series for $(t^{2k-1}-2t^{2k})(1-4t)^{-\frac{3}{2}}$ on the right hand side of (3.5). Using this with (3.8), we obtain

$$C_n r_n^\tau \sim \frac{1}{2}(n-2k+2)(n-2k+3)C_{n-2k+2}-(n-2k+1)(n-2k+2)C_{n-2k+1}. \tag{3.11}$$

Since $\lim_{n\to\infty}\frac{C_{n-1}}{C_n}=\frac{1}{4}$, it follows from (3.11) that

$$E_n^{\text{av}(231)}(N_n^{(k;\tau)})^2 = r_n^\tau \sim n^2\big(\frac{1}{2}4^{2-2k}-4^{1-2k}\big) = \frac{n^2}{4^{2k-1}}. \tag{3.12}$$

From (1.9), we have

$$E_n^{\text{av}(231)}N_n^{(k;\tau)} \sim \frac{1}{2\cdot 4^{k-1}}. \tag{3.13}$$

Thus, it follows from (3.12) and (3.13) that

$$\text{Var}(N_n^{(k;\tau)}) = o(n^2).$$

□

4. Proof of Theorem 2

As noted in the introduction, it suffices to consider the case $\eta = 321$. The key step to proving the theorem is the following result, whose rather technical proof will be postponed until after the proof of Theorem 2. Recall from the introduction that $A_n^{k;l}\subset S_n$ denotes the event that the set of k consecutive numbers $\{l,l+1,\cdots,l+k-1\}$ appears in a set of consecutive positions. For $\tau\in S_k$, let $A_n^{k,\tau;l}\subset S_n$ denote the event that the set of k consecutive numbers $\{l,l+1,\cdots,l+k-1\}$ appears in a set of consecutive positions according to the pattern τ.

Proposition 5. *For $n \ge k \ge 2$ and $1 \le l \le n-k+1$,*

$$P_n^{av(321)}(A_n^{k,\tau;l}) = \begin{cases} \frac{C_{l-1}C_{n-k-l+1}}{C_n}, \ \textit{if } 1\cdots k \ne \tau \in S_k^{av(321)}; \\ \frac{C_{n-k+1}}{C_n}, \ \textit{if } \tau = 1\cdots k. \end{cases} \tag{4.1}$$

The proof of Theorem 2 is almost immediate from Proposition 5.

Proof of Theorem 2. Since $N_n^{(k;\tau)} = \sum_{l=1}^{n-k+1} 1_{A_n^{k,\tau;l}}$, Proposition 5 yields

$$E_n^{\text{av}(321)} N_n^{(k,\tau)} = \sum_{l=1}^{n-k+1} P_n^{\text{av}(321)}(A_n^{k,\tau;l}) = \begin{cases} \frac{C_{n-k+1}}{C_n}, \ \text{if } 1\cdots k \ne \tau \in S_k^{\text{av}(321)}; \\ \frac{(n-k+1)C_{n-k+1}}{C_n}, \ \text{if } \tau = 1\cdots k, \end{cases}$$

where the latter equality in the case $\tau \ne 1\cdots k$ follows from (2.2). This gives (1.13) in the case $\eta = 321$. Since there are $C_k - 1$ permutations $1\cdots k \ne \tau \in S_k^{\text{av}(321)}$, (1.14) follows from (1.13).

Proof of Proposition 5. Fix n, k, l as in the statement of the proposition. Fix $\tau \in S_k^{\text{av}(321)}$. For $1 \le a \le n-k+1$, define

$$A_n^{k,\tau;l,a} = \{\sigma \in A_n^{k,\tau;l} : \{l, l+1, \cdots, l+k-1\} = \{\sigma_a, \sigma_{a+1}, \cdots \sigma_{a+k-1}\}.$$

Then the sets $\{A_n^{k,\tau;l,a}\}_{a=1}^{n-k+1}$ are disjoint and $A_n^{k,\tau;l} = \cup_{a=1}^{n-k+1} A_n^{k,\tau;l;a}$.

If $\nu = \{\nu_i\}_{i=1}^{|B|}$ is a permutation of a finite set $B \subset \mathbb{N}$, let $\nu^{B^{-1}}$ denote the permutation it naturally induces on $S_{|B|}$; that is, $\nu^{B^{-1}} = \text{red}(\nu)$, where $\text{red}(\cdot)$ was defined at the beginning of section 2. Conversely, if ν is a permutation of $S_{|B|}$, let ν^B denote the permutation it naturally induces on B.

Until further notice, consider a fixed. Let $\sigma \in A_n^{k,\tau;l;a} \cap S_n^{\text{av}(321)}$. We describe a procedure to contract σ to a permutation in $S_{n-k+1}^{\text{av}(321)}$. Define the permutation $\overline{\sigma} = \overline{\sigma}(\sigma)$ of the set $B = \{1, \cdots, l, l+k, \cdots, n\}$ by

$$\overline{\sigma}_i = \begin{cases} \sigma_i, \ 1 \le i \le a-1; \\ l, \ i = a; \\ \sigma_{i+k-1}, \ i = a+1, \cdots, n-k+1, \end{cases}$$

and define

$$\nu = \nu(\sigma) = \overline{\sigma}^{B^{-1}}.$$

It follows from the construction that

$$\nu \in S_{n-k+1}^{\text{av}(321)} \text{ and } \nu_a = l. \tag{4.2}$$

We concretize the above construction with an example. Let $n = 9, k = 3, l = a = 4$. Let $\sigma = 213546897 \in A_9^{3,213;4;4} \cap S_9^{\text{av}(321)}$. The set B is given by $B = \{1,2,3,4,7,8,9\}$ and $\bar{\sigma} = 2134897$—the cluster 546 in σ has been contracted to 4 in $\bar{\sigma}$. Finally, $\nu = \nu(\sigma) = \bar{\sigma}^{B^{-1}} = 2134675$ satisfies $\nu \in S_7^{\text{av}(321)}$ and $\nu_4 = 4$.

Obviously the map taking $\sigma \in A_n^{k,\tau;l;a} \cap S_n^{\text{av}(321)}$ to $\nu(\sigma)$ is not injective. However,

$$\begin{aligned} &\nu(\sigma) \neq \nu(\sigma'), \text{ if } \sigma, \sigma' \in A_n^{k,\tau;l;a} \cap S_n^{\text{av}(321)} \text{ are distinct and satisfy} \\ &\sigma_{a+i} = \sigma'_{a+i}, \ i = 0, \cdots, k-1. \end{aligned} \tag{4.3}$$

Conversely, let ν satisfy (4.2). We describe a procedure to extend ν to a permutation in $A_n^{k,\tau;l;a}$, which may or may not belong to $S_n^{\text{av}(321)}$. Let $B = \{1, \cdots, l, l+k, \cdots, n\}$ as above. Define $\sigma^\tau = \sigma^\tau(\nu) \in S_n$ by

$$\sigma_i^\tau = \begin{cases} \nu_i^B, \ i = 1, \cdots, a-1; \\ l - 1 + \tau_{i-a+1}, \ i = a, \cdots, a+k-1; \\ \nu_{i-k+1}^B, \ i = a+k, \cdots, n. \end{cases}$$

It follows from the construction that

$$\sigma^\tau(\nu) \in A_n^{k,\tau;l;a}. \tag{4.4}$$

Also, of course, the map taking ν satisfying (4.2) to $\sigma^\tau(\nu)$ is injective.

As an example of the above construction, again with $n = 9, k = 3, l = a = 4$, let $\nu = 2134675$. Then ν satisfies (4.2). We have $B = \{1,2,3,4,7,8,9\}$. Choose, for example, $\tau = 213 \in S_3$. Then $\nu^B = 2134897$ and $\sigma^\tau = \sigma^\tau(\nu) = 213546897 \in A_9^{3,213;4,4}$—the 4 in ν^B has been expanded to the cluster 546 in σ^τ.

We now investigate when in fact $\sigma^\tau(\nu) \in S_n^{\text{av}(321)}$, or equivalently in light of (4.4), when $\sigma^\tau(\nu) \in S_n^{\text{av}(321)} \cap A_n^{k,\tau;l;a}$. If $\tau = 12\cdots k$, then it is clear that $\sigma^\tau(\nu) \in S_n^{\text{av}(321)}$, for all ν satisfying (4.2). Thus, since the map taking

$\sigma \in A_n^{k,\tau;l;a} \cap S_n^{\text{av}(321)}$ to $\nu(\sigma)$ satisfies (4.3), and the map taking ν satisfying (4.2) to $\sigma^\tau(\nu)$ is injective, it follows that

$$|S_n^{\text{av}(321)} \cap A_n^{k,\tau;l;a}| = |\{\nu \in S_{n-k+1}^{\text{av}(321)} : \nu_a = l\}|, \ \tau = 1 \cdots k. \tag{4.5}$$

Now consider any of the other $\tau \in S_k^{\text{av}(321)}$. Since τ has a decreasing subsequence of length 2, in order to have $\sigma^\tau(\nu) \in S_n^{\text{av}(321)}$, all of the numbers $\{1, \cdots, l-1\}$ must appear among the first $a-1$ positions of ν, and all the numbers $\{l+1, \cdots n-k+1\}$ must appear among the last $n-a-k+1$ positions of ν. This is possible only if $a = l$. If indeed $a = l$, then $\sigma^\tau(\nu) \in S_n^{\text{av}(321)}$ if and only if the first $l-1$ positions of ν are filled in a 321-avoiding way by the numbers $\{1, \cdots, l-1\}$ and the last $n-l-k+1$ positions of ν are filled in a 321-avoiding way by the numbers $\{l+1, \cdots n-k+1\}$. (The one remaining position, position a, is by assumption filled by the number l.) Thus, again because the map taking $\sigma \in A_n^{k,\tau;l;a} \cap S_n^{\text{av}(321)}$ to $\nu(\sigma)$ satisfies (4.3), and the map taking ν satisfying (4.2) to $\sigma^\tau(\sigma)$ is injective, it follows that

$$|S_n^{\text{av}(321)} \cap A_n^{k,\tau;l;a}| = \begin{cases} C_{l-1}C_{n-l-k+1}, \ a = l; \\ 0, \ a \neq l. \end{cases} \quad \text{for } 1 \cdots k \neq \tau \in S_k^{\text{av}(321)}. \tag{4.6}$$

Summing (4.5) and (4.6) over $a \in \{1, \cdots, n-k+1\}$, we obtain

$$|S_n^{\text{av}(321)} \cap A_n^{k,\tau;l}| = \begin{cases} C_{n-k+1}, \ \tau = 1 \cdots k; \\ C_{l-1}C_{n-l-k+1}, \ 1 \cdots k \neq \tau \in S_k^{\text{av}(321)}. \end{cases} \tag{4.7}$$

From (4.7), we conclude that (4.1) holds. □

5. Proof of Theorem 3

With just one change, we follow the construction appearing in the proof of Proposition 5 above, from the beginning of the proof up until but not including the paragraph containing (4.5). The one change is that wherever $S_m^{\text{av}(321)}$ appears, for some $m \in \mathbb{N}$, it needs to be replaced by $S_m^{\text{av}(\eta_1, \cdots, \eta_r)}$. (Therefore, the two examples appearing in the construction also needed to be amended.) Thus, in the sequel, whenever we refer to equations appearing in the above noted construction, any appearance of $S_m^{\text{av}(321)}$ in such an equation must be changed to $S_m^{\text{av}(\eta_1,\cdots,\eta_r)}$. As in the proof of Proposition 5, we now investigate when in fact $\sigma^\tau(\nu) \in S_n^{\text{av}(\eta_1,\cdots,\eta_r)}$, or equivalently in light of (4.4), when $\sigma^\tau(\nu) \in S_n^{\text{av}(\eta_1,\cdots,\eta_r)} \cap A_n^{k,\tau;l;a}$. In fact, this holds for all $\tau \in S_k^{\text{av}(\eta_1,\cdots,\eta_r)}$. Indeed, since by (4.2), $\nu \in S_{n-k+1}^{\text{av}(\eta_1,\cdots,\eta_r)}$, and since $\tau \in S_k^{\text{av}(\eta_1,\cdots,\eta_r)}$, it follows from the definition of $\sigma^\tau(\nu)$ that if $\sigma^\tau(\nu) \notin S_n^{\text{av}(\eta_1,\cdots,\eta_r)}$, then for some $i \in [r]$ and some $2 \le j_0 \le k-1$, $\sigma^\tau(\nu)$ contains the pattern η_i and exactly j_0 of the numbers in $\{\sigma_a^\tau(\nu), \cdots, \sigma_{a+k-1}^\tau(\nu)\} = \{l, \cdots, l+k-1\}$ are used in the construction of the pattern η_i. But then it would follow that η_i has a nontrivial block of length j_0, which contradicts the assumption that η_i is simple.

Since for all $\tau \in S_k^{\text{av}(\eta_1,\cdots,\eta_r)}$, we have $\sigma^\tau(\nu) \in S_n^{\text{av}(\eta_1,\cdots,\eta_r)} \cap A_n^{k,\tau;l;a}$, and since the map taking $\sigma \in A_n^{k,\tau;l;a} \cap S_n^{\text{av}(\eta_1,\cdots,\eta_r)}$ to $\nu(\sigma)$ satisfies (4.3), and the map taking ν satisfying (4.2) to $\sigma^\tau(\nu)$ is injective, it follows that

$$|S_n^{\text{av}(\eta_1,\cdots,\eta_r)} \cap A_n^{k,\tau;l;a}| = |\{\nu \in S_{n-k+1}^{\text{av}(\eta_1,\cdots,\eta_r)} : \nu_a = l\}|. \tag{5.1}$$

Summing (5.1) over $a \in \{1, \cdots, n-k+1\}$, we obtain

$$|S_n^{\text{av}(\eta_1,\cdots,\eta_r)} \cap A_n^{k,\tau;l}| = |S_{n-k+1}^{\text{av}(\eta_1,\cdots,\eta_r)}|. \tag{5.2}$$

From (5.2), we obtain

$$P_n^{\text{av}(\eta_1,\cdots,\eta_r)}(A_n^{k,\tau;l}) = \frac{|S_{n-k+1}^{\text{av}(\eta_1,\cdots,\eta_r)}|}{|S_n^{\text{av}(\eta_1,\cdots,\eta_r)}|}. \tag{5.3}$$

(We note that this construction leading to (5.3) is similar to a construction in [1].) Since

$$E_n^{\text{av}(\eta_1,\cdots,\eta_r)} N^{(k;\tau)} = \sum_{l=1}^{n-k+1} P_n^{\text{av}(\eta_1,\cdots,\eta_r)}(A_n^{k,\tau;l}),$$

(1.17) follows from (5.3), and then (1.18) follows by summing (1.17) over $\tau \in S_k^{\text{av}(\eta_1,\cdots,\eta_r)}$. □

References

[1] Albert, M. H. and Atkinson, M. D., *Simple permutations and pattern restricted permutations. Discrete Math.* **300** (2005), 1-15.

[2] Atapour, M. and Madras, N., *Large deviations and ratio limit theorems for pattern-avoiding permutations*, Combin. Probab. Comput. **23** (2014), 161-200.

[3] Bóna, M., *On three different notions of monotone subsequences*, Permutation patterns, 89-114, London Math. Soc. Lecture Note Ser., **376**, Cambridge Univ. Press, Cambridge, (2010).

[4] Bose, P., Buss, J. and Lubiw, A., *Pattern matching for permutations*, Inform. Process. Lett. **65** (1998), 277-283.

[5] Flajolet, P. and Sedgewick, R., *Analytic Combinatorics*, Cambridge University Press, Cambridge, (2009).

[6] Kaplansky, I., *The asymptotic distribution of runs of consecutive elements*, Ann. Math. Statistics **16** (1945), 200-203.

[7] Marcus, A. and Tardos, G., *Excluded permutation matrices and the Stanley-Wilf conjecture*, J. Combin. Theory Ser. A **107** (2004), 153-160.

[8] Pinsky, R. *Problems from the discrete to the continuous. Probability, number theory, graph theory, and combinatorics*, Universitext. Springer, Cham, (2014).

[9] Pinsky, R. *The infinite limit of separable permutations*, Random Structures Algorithms **59** (2021), 622-639,

[10] Pinsky, R., *Clustering of consecutive numbers in permutations under Mallows distributions and super-clustering under general p-shifted distributions*, Electron. J. Probab. **27** (2022), Paper No. 87, 20 pp.

[11] Simion, R. and Schmidt, F.W. *Restricted permutations*, European J. Combin. **6** (1985), 383-406.

[12] Wolfowitz, J. *Note on runs of consecutive elements*, Ann. Math. Statistics **15** (1944), 97-98.

Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

Email address: `pinsky@math.technion.ac.il`

URL: `https://pinsky.net.technion.ac.il/`