MEAN AND VARIANCE OF THE LONGEST ALTERNATING SUBSEQUENCE IN A RANDOM SEPARABLE PERMUTATION

ROSS G. PINSKY

Abstract. A permutation is *separable* if it can be obtained from the singleton permutation by iterating direct sums and skew sums. Equivalently, it is separable if and only it avoids the patterns 2413 and 3142. Under the uniform probability on separable permutations of $[n]$, let the random variable A_n denote the length of the longest alternating subsequence. Also, let $A_n^{+,-}$ denote the length of the longest alternating subsequence that begins with an ascent and ends with a descent, and define $A_n^{-,+}, A_n^{+,+}, A_n^{-,-}$ similarly. By symmetry, the first two and the last two of these latter four random variables are equi-distributed. We prove that the expected value of any of these five random variables behaves asymptotically as $(2-\sqrt{2})n \approx 0.5858\,n$. We also obtained the more refined estimates that the expected value of $A_n^{+,-}$ and of $A_n^{-,+}$ is equal to $(2-\sqrt{2})n - \frac{1}{4}(3-2\sqrt{2}) + o(1)$ and that the expected value of $A_n^{+,+}$ and of $A_n^{-,-}$ is equal to $(2-\sqrt{2})n + \frac{3}{4}(3-2\sqrt{2}) + o(1)$. Finally, we show that the variance of any of the four random variables $A_n^{\pm,\pm}$ behaves asymptotically as $\frac{16-11\sqrt{2}}{2}n \approx 0.2218\,n$.

1. Introduction and Statement of Results

Let S_n denote the permutations of $[n] := \{1, \cdots, n\}$. Given $\sigma \in S_k$ and $\tau \in S_l$, the *direct sum* of σ and τ is the permutation in S_{k+l} given by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i), \ i = 1, \cdots, k; \\ \tau(i-k) + k, \ i = k+1, \cdots k+l, \end{cases}$$

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and the *skew sum* $\sigma \ominus \tau$ is the permutation in S_{k+l} given by

$$(\sigma \ominus \tau)(i) = \begin{cases} \sigma(i) + l, \ i = 1, \cdots, k; \\ \tau(i-k), \ i = k+1, \cdots k+l. \end{cases}$$

A permutation is *indecomposable* if it cannot be represented as the direct sum of two nonempty permutations and is *skew indecomposable* if it cannot be represented as the skew sum of two nonempty permutations. A permutation is *separable* if it can be obtained from the singleton permutation by iterating direct sums and skew sums. Equivalently, a permutation is separable if it can be successively decomposed and skew decomposed until all of the indecomposable and skew indecomposable pieces of the permutation are singletons. For example, using one-line notation, consider the separable permutation $\sigma = 4352167$. It can be decomposed into $43521 \oplus 12$. Then 43521 can be skew decomposed into $213 \ominus 21$ and 12 can be decomposed into $1 \oplus 1$. Now 213 can be decomposed into $21 \oplus 1$ and 21 can be skew decomposed into $1 \ominus 1$. Finally, again 21 can be skew decomposed into $1 \ominus 1$.

It is well-known [3] that a permutation is separable if and only it avoids the patterns 2413 and 3142. For more on pattern avoiding permutations, see for example [2]. The fact that separable permutations can be enumerated by a closed form generating function (see section 2) makes them rather tractable to analyze. The study of general pattern avoiding permutations goes back to Knuth's observation [7] that a permutation is so-called stack sortable if and only if it 231-avoiding. Similarly, the study of separable permutations goes back to [1] where it was shown that these are precisely the permutations which are sortable by so-called pop stacks. Separable permutations also arise in a variety of other applications, for example in bootstrap percolation [8] and in connection to polynomial interchanges where one studies the possible ways that the relative order of the values of a family of polynomials can be modified when crossing a common zero [6].

Let $\text{SEP}(n)$ denote the set of separable permutations in S_n, and let P_n^{sep} and E_n^{sep} denote respectively the uniform probability measure on $\text{SEP}(n)$ and the expectation with respect to that measure. In this paper we study the length of the longest alternating subsequence in a random separable permutation. An alternating subsequence of length k in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is a subsequence of the form $\sigma_{i_1} > \sigma_{i_2} < \sigma_{i_3} > \cdots \sigma_{i_k}$ or

$\sigma_{i_1} < \sigma_{i_2} > \sigma_{i_3} < \cdots \sigma_{i_k}$, where $1 \le i_1 < \cdots < i_k \le n$. Let $A_n = A_n(\sigma)$ denote the length of the longest alternating subsequence in a permutation $\sigma \in S_n$. For the proofs of our results as well as for the statements of some of them, we need to differentiate between four types of alternating sequences—begin with an ascent and end with an ascent; begin with an ascent and end with a descent; begin with a descent and end with an ascent; begin with a descent and end with a descent. An alternating subsequence of length k which begins with an ascent and ends with an ascent in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is a subsequence of the form $\sigma_{i_1} < \sigma_{i_2} > \cdots < \sigma_{i_k}$. The other three types are defined similarly. Denote the length of the longest alternating subsequence of each of the four types of alternating subsequences by $A_n^{\pm,\pm}$. From symmetry considerations, it is clear that

$$A_n^{+,+} \stackrel{\text{dist}}{=} A_n^{-,-}; \quad A_n^{+,-} \stackrel{\text{dist}}{=} A_n^{-,+}, \text{ under } P_n^{\text{sep}}. \tag{1.1}$$

Stanley [9] obtained exact formulas for the expected value and the variance of $A_n^- := \max(A_n^{-,-}, A_n^{-,+})$ for a uniformly random permutation in S_n, from which it follows that the expectation is asymptotic to $\frac{2}{3}n$ and the variance is asymptotic to $\frac{8}{45}n$. The length of the longest alternating subsequence in a random permutation avoiding a pattern of length three was studied in [4]. Exact formulas were obtained for the expected value and the variance of $A_n^+ := \max(A_n^{+,+}, A_n^{+,-})$, for each pattern $\eta \in S_3$, from which it follows that the expectation is asymptotic to $\frac{1}{2}n$ and the variance is asymptotic to $\frac{1}{4}n$, for every pattern $\eta \in S_3$.

We will prove the following theorem concerning the expectation.

Theorem 1.

$$E_n^{\text{sep}} A_n \sim (2 - \sqrt{2})n \approx .5858\, n. \tag{1.2}$$

Remark 1. Since $0 \le A_n - A_n^{\pm,\pm} \le 2$, (1.2) also holds when any of the four random variables $A_n^{\pm,\pm}$ is substituted for A_n.

Remark 2. Note that the expected length of the longest alternating subsequence in a random separable permutation is shorter than that of a uniformly random permutation, but longer than that of a random permutation avoiding any particular pattern of length three. (See the paragraph preceding Theorem 1.)

We will prove the following theorem which refines Theorem 1 and gives the asymptotic behavior of the variance.

Theorem 2. *i.*

$$E_n^{\text{sep}}A_n^{+,-}=E_n^{\text{sep}}A_n^{-,+}=(2-\sqrt{2})n-\frac{1}{4}(3-2\sqrt{2})+o(1); \tag{1.3}$$

$$\text{Var}_n^{\text{sep}}(A_n^{+,-})=\text{Var}_n^{\text{sep}}(A_n^{-,+})\sim\frac{16-11\sqrt{2}}{2}n\approx 0.2218\,n. \tag{1.4}$$

ii.

$$E_n^{\text{sep}}A_n^{+,+}=E_n^{\text{sep}}A_n^{-,-}=(2-\sqrt{2})n+\frac{3}{4}(3-2\sqrt{2})+o(1); \tag{1.5}$$

$$\text{Var}_n^{\text{sep}}(A_n^{+,+})=\text{Var}_n^{\text{sep}}(A_n^{-,-})\sim\frac{16-11\sqrt{2}}{2}n\approx 0.2218\,n. \tag{1.6}$$

Remark 1. From Theorem 2, it follows that from the limited perspective of mean and variance, $A_n^{+,+}$ or $A_n^{-,-}$ behaves as a deterministic translation of $A_n^{+,-}$ or $A_n^{-,+}$ by $3-2\sqrt{2}$. We note that the quantity $3-2\sqrt{2}$ plays a fundamental role in the proofs; it is one of the roots of the generating function corresponding to the count of separable permutations–see section 2.

Remark 2. By Theorems 1 and 2, one has $\text{Var}_n^{\text{sep}}(A_n^{\pm,\pm})=o\left(\left(E_n^{\text{sep}}A_n^{\pm,\pm}\right)^2\right)$, for all four random variables $A_n^{\pm,\pm}$. Thus, by the second moment method (Chebyshev's inequality), the following weak law of large numbers is immediate:

$$\lim_{n\to\infty}P_n^{\text{sep}}\left(2-\sqrt{2}-\epsilon\le\frac{A_n^{\pm,\pm}}{n}\le 2-\sqrt{2}+\epsilon\right)=1,\ \text{for all } \epsilon>0.$$

In section 2 we present some preliminary material on separable permutations and on alternating subsequences. In section 3, we define two generating functions related to the mean of the length of the longest alternating sequence and evaluate them explicitly. Using one of these generating functions, we prove Theorem 1 in section 4. In section 5, we define two generating functions related to the second moment of the length of the longest alternating sequence and evaluate them explicitly. Using all four of the above noted generating functions, we prove Theorem 2 in section 6. In the appendix we state and prove three propositions, concerning the asymptotic behavior as $n \to \infty$ of the coefficient of t^n in the power series for the functions $(t^2 - 6t + 1)^{\frac{m}{2}}$, for $m \in \{1, -1, -3\}$. These results are critical for the proofs of Theorems 1 and 2. More precisely, formulas (7.1) from Proposition 1 and (7.28) from Proposition 2 are needed for the proof of Theorem 1, while their more precise versions, (7.2) and (7.29), along with formula (7.33) in Proposition 3 are needed for the proof of Theorem 2.

2. Preliminary material

Let $s_n = |\mathrm{SEP}(n)|,\ n \geq 1$, denote the number of separable permutations in S_n. Let

$$s(t) = \sum_{n=1}^{\infty} s_n t^n$$

denote the generating function of $\{s_n\}_{n=1}^{\infty}$. For a separable permutation, define the length of the first indecomposable block and the length of the first skew indecomposable block respectively by

$$\begin{aligned} &B_1^{+,n}(\sigma) = \min\{j : \sigma([j]) = [j]\},\ \sigma \in \mathrm{SEP}(n); \\ &B_1^{-,n}(\sigma) = \min\{k : \sigma([k]) = [n] - [n-k]\}, \sigma \in \mathrm{SEP}(n). \end{aligned} \tag{2.1}$$

By the definition of separable permutations, for each $\sigma \in \mathrm{SEP}(n)$, with $n \geq 2$, exactly one out of $B_1^{+,n}(\sigma)$ and $B_1^{-,n}(\sigma)$ is equal to n, and by symmetry,

$$|\{\sigma \in \mathrm{SEP}(n) : B_1^{+,n}(\sigma) = n| = |\{\sigma \in \mathrm{SEP}(n) : B_1^{-,n}(\sigma) = n| = \frac{1}{2} s_n,\ n \geq 2. \tag{2.2}$$

That is, half of the permutations in $\mathrm{SEP}(n)$, $n \geq 2$, are indecomposable and half are skew indecomposable. Partitioning $\mathrm{SEP}(n)$ by $\{B_1^{+,n} = j\}_{j=1}^n$ (or alternatively, by $\{B_1^{-,n} = j\}_{j=1}^n$), and using the concatenating structure of separable permutations, it follows that

$$s_n = s_1 s_{n-1} + \frac{1}{2} \sum_{j=2}^{n-1} s_j s_{n-j} + \frac{1}{2} s_n,\ n \geq 2, \tag{2.3}$$

and

$$P_n^{\mathrm{sep}}(B_1^{+,n} = j) = P_n^{\mathrm{sep}}(B_1^{-,n} = j) = \begin{cases} \frac{s_1 s_{n-1}}{s_n}; j = 1; \\ \frac{\frac{1}{2} s_j s_{n-j}}{s_n}, j = 2 \cdots n-1; \\ \frac{1}{2}, j = n. \end{cases} \tag{2.4}$$

From (2.3) it is straightforward to show that

$$s(t)=\frac{1}{2}(1-t-\sqrt{t^2-6t+1}\,),\ \text{for } |t|<3-2\sqrt{2}. \tag{2.5}$$

(Multiply both sides of (2.3) by t^n and sum over n from 2 to ∞, and then solve for $s(t)$.) Using the above formula for the generating function, one can prove that

$$s_n\sim\frac{1}{2^{\frac34}\sqrt{\pi n^3}}(3-2\sqrt{2})^{-n+\frac12}.$$

We prove this asymptotic formula as well as a more refined version in Proposition 1. We note that in [5, p. 474-475], the above formula appears with a mistake—instead of $2^{\frac34}$, one find there 2. (Our s_n is equal to their D_{n-1}.) The sequence of integers $\{s_n\}_{n=1}^\infty$ is known as the sequence of big Schröder numbers; see A006318 in the *On-Line Encyclopedia of Integer Sequences.*

As noted before the statement of Theorem 2, we differentiate between four types of alternating sequences, depending on whether they begin with an ascent or a descent and whether they end with an ascent or a descent, and denote the length of the longest one of each type by $A_n^{\pm,\pm}$. We note that a subsequence of length two of the form $\sigma_{i_1}<\sigma_{i_2}$ begins with an ascent and ends with an ascent. We will derive recursion formulas that allow us to obtain generating functions in explicit forms. In order to make the formulas work, the following definition will be crucial:

(2.6)

A singleton σ_{i_1} is considered an alternating sequence both of the type $(+,-)$ and of the type $(-,+)$;

A singleton σ_{i_1} is not considered an alternating sequence of the type $(+,+)$ or of the type $(-,-)$.

In light of (2.6), note that

$$\begin{aligned}&A_1^{+,+}=A_1^{-,-}\equiv 0;\quad A_1^{+,-}=A_1^{-,+}\equiv 1;\\&A_2^{+,-}=A_2^{-,+}\equiv 1\\&A_2^{+,+}(\sigma)=\begin{cases}2, & \sigma=12,\\ 0, & \sigma=21;\end{cases}\qquad A_2^{-,-}(\sigma)=\begin{cases}0, & \sigma=12,\\ 2, & \sigma=21.\end{cases}\end{aligned} \tag{2.7}$$

Remark. Note that the definition (2.6) ensures that $A_n^{+,+}$ and $A_n^{-,-}$ take on only even values and that $A_n^{+,-}$ and $A_n^{-,+}$ take on only odd values.

The following lemma will play an important role. Let $P_n^{\text{sep}}(\cdot\ |B_1^{+;n}=n)$ $\big(P_n^{\text{sep}}(\cdot\ |B_1^{-;n}=n)\big)$ denote the distribution P_n^{sep} conditioned on $\{B_1^{+;n}=n\}$ $\big(\{B_1^{-;n}=n\}\big)$.

Lemma 1. *i. The distribution of* $A_n^{+,+}$ $\big(A_n^{-,-}\big)$ *under* $P_n^{\text{sep}}(\cdot\ |B_1^{+;n}=n)$ *coincides with the distribution of* $A_n^{-,-}$ $\big(A_n^{+,+}\big)$ *under* $P_n^{\text{sep}}(\cdot\ |B_1^{-;n}=n)$*;*
ii. The distribution of $A_n^{+,-}$ $\big(A_n^{-,+}\big)$ *under* $P_n^{\text{sep}}(\cdot\ |B_1^{+;n}=n)$ *coincides with the distribution of* $A_n^{-,+}$ $\big(A_n^{+,-}\big)$ *under* $P_n^{\text{sep}}(\cdot\ |B_1^{-;n}=n)$*;*
iii. The distributions of $A_n^{+,-}$ *and* $A_n^{-,+}$ *coincide under* $P_n^{\text{sep}}(\cdot\ |B_1^{+;n}=n)$*.*

Proof. Recall that the *reverse* of a permutation $\sigma=\sigma_1\cdots\sigma_n$ is the permutation $\sigma^{\text{rev}}:=\sigma_n\cdots\sigma_1$, and the *complement* of σ is the permutation σ^{com} satisfying $\sigma_i^{\text{com}}=n+1-\sigma_i,\ i=1,\cdots,n$. Let $\sigma^{\text{rev-com}}$ denote the permutation obtained by applying reversal and then complementation to σ (or equivalently, vice versa). All of these operations are bijections of SEP(n); indeed, they all preserve the property of being simultaneously 2413 and 3142 avoiding. It is easy to see that

$$B_1^{+,n}(\sigma)=1\ \Leftrightarrow\ B_1^{-,n}(\sigma^{\text{rev}})=n\ \Leftrightarrow\ B_1^{-,n}(\sigma^{\text{com}})=n,\ \sigma\in S_n.$$

From the above facts, it follows that

(2.8)
$$\begin{aligned}&\sigma\to\sigma^{\text{com}}\text{ is a bijection of SEP}(n)\cap\{B_1^{+,n}=n\}\text{ to SEP}(n)\cap\{B_1^{-,n}=n\};\\&\sigma\to\sigma^{\text{rev-com}}\text{ is a bijection of SEP}(n)\cap\{B_1^{+,n}=n\}\text{ to itself.}\end{aligned}$$

Also, it is easy to check that

$$\begin{aligned}A_n^{+,+}(\sigma)&=A_n^{-,-}(\sigma^{\text{com}}),\ \sigma\in S_n;\\A_n^{+,-}(\sigma)&=A_n^{-,+}(\sigma^{\text{com}});\\A_n^{+,-}(\sigma)&=A_n^{+,-}(\sigma^{\text{rev}}).\end{aligned}\tag{2.9}$$

From the latter two equations in (2.9), it follows that

$$A_n^{+,-}(\sigma)=A_n^{-,+}(\sigma^{\text{rev-com}}).\tag{2.10}$$

Part (i) of the lemma follows from the first line of (2.8) and the first line of (2.9); part (ii) follows from the first line of (2.8) and the second line of (2.9); and part (iii) follows from the second line of (2.8) and (2.10). $\square$

3. Generating functions related to the mean of A_n

Define

$$c_n^{\pm,\pm}=E_n^{\mathrm{sep}}(A_n^{\pm,\pm}|B_1^{+,n}=n);\quad d_n^{\pm,\pm}=E_n^{\mathrm{sep}}(A_n^{\pm,\pm}|B_1^{-,n}=n). \tag{3.1}$$

Define the generating functions

$$\begin{aligned}G^{+,-}(t)&=\sum_{n=1}^{\infty}s_nc_n^{+,-}t^n;\\ G^{-,-}(t)&=\sum_{n=1}^{\infty}s_nc_n^{-,-}t^n.\end{aligned} \tag{3.2}$$

We will prove the following theorem.

Theorem 3.

$$\begin{aligned}G^{+,-}(t)&=\frac{t(1-t)}{\sqrt{t^2-6t+1}};\\ G^{-,-}(t)&=\frac{t(1-t)}{\sqrt{t^2-6t+1}}-\frac{t(1-t)}{2}-\frac{t}{2}\sqrt{t^2-6t+1}.\end{aligned} \tag{3.3}$$

Proof. By (2.2) we have

$$E_n^{\mathrm{sep}}A_n^{\pm,\pm}=\frac{1}{2}c_n^{\pm,\pm}+\frac{1}{2}d_n^{\pm,\pm}. \tag{3.4}$$

Conditioning on $\{B_1^{+,n}=j\}, j=1,\cdots,n$, we obtain the equation

$$E_n^{\mathrm{sep}}A_n^{\pm,\pm}=\sum_{j=1}^{n}P_n^{\mathrm{sep}}(B_1^{+,n}=j)E\mathrm{sep}_n(A_n^{\pm,\pm}|B_1^{+,n}=j). \tag{3.5}$$

Consider (3.5) with $A_n^{+,+}$. From the definition of $A_n^{+,+}$ and the concatenating structure of separable permutations as manifested in (2.3), we have

(3.6)

$$A_n^{+,+}|\{B_1^{+,n}=j\}\stackrel{\mathrm{dist}}{=}A_j^{+,-}|\{B_1^{+,j}=j\}+A_{n-j}^{-,+},\ j=1,\cdots,n-1,\ n\geq 3.$$

where the random variable on the left hand side is considered under P_n^{sep}, the random variable $A_j^{+,-}|\{B_1^{+,j}=j\}$ is considered under P_j^{sep}, the random variable $A_{n-j}^{-,+}$ is considered under P_{n-j}^{sep}, and $A_j^{+,-}|\{B_1^{+,j}=j\}$ and $A_{n-j}^{-,+}$ are independent.

To illustrate (3.6), we give three examples of what can occur. Consider first the permutation 342178956, which satisfies $B_1^{+,9}(342178956)=4$. So $n=9$, $j=4$ and $n-j=5$. The length of the longest alternating subsequence that begins and ends with an ascent is six. Such an alternating subsequence is built from a longest alternating subsequence beginning with an ascent and ending with a descent that appears in the first j entries of the permutation— 3421, and then concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n-j$ entries of the permutation— 78956. There are two possibilities for the first piece, namely 342 and 341, and there are three possibilities for the second piece, namely 756 and 856 and 956. Consider now the permutation 124378956, which satisfies $B_1^{+,9}(124378956) = 1$. So $n = 9$, $j = 1$ and $n-j = 8$. The length of the longest alternating subsequence that begins and ends with an ascent is six. Such an alternating subsequence is built from a longest alternating subsequence beginning with an ascent and ending with a descent that appears in the first j entries of the permutation— 1, and concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n-j$ entries of the permutation— 24378956. There is one possibility for the first piece, namely 1, and there are three possibilities for the second piece, namely 43756, 43856 and 43956. Note that for this to work, it was necessary in (2.6) that a singleton be defined as an alternating sequence of the type $(+,-)$. Finally, consider the permutation 324561789, which satisfies $B_1^{+,9}(324561789) = 6$. So $n = 9$, $j = 6$ and $n-j = 3$. The length of the longest alternating subsequence that begins and ends with an ascent is four. Such an alternating subsequence is built from a longest alternating subsequence beginning with an ascent and ending with a descent that appears in the first j entries of the permutation— 324561, and then concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n-j$ entries of the permutation— 789. There are six possibilities for the first piece, namely 341,351,361,241,251,261, and there are three possibilities for the second piece, namely 7, 8, 9. Note that for this to work, it was necessary in (2.6) that a singleton be defined as an alternating sequence of the type $(-,+)$.

Now consider (3.5) with $A_n^{-,+}$. From the definition of $A_n^{-,+}$ and the concatenating structure of separable permutations as manifested in (2.3), we have

(3.7)

$$A_n^{-,+}|\{B_1^{+,n}=j\} \stackrel{\text{dist}}{=} A_j^{-,-}|\{B_1^{+,j}=j\}+A_{n-j}^{-,+},\ j=1,\cdots,n-1,\ n\ge 3.$$

To illustrate (3.7), we give two examples of what can occur. Consider first the permutation 342178956, which satisfies $B_1^{+,n}(342178956)=4$. So $n=9, j=4$ and $n-j=5$. The length of the longest alternating subsequence that begins with a descent and ends with an ascent is five. Such an alternating subsequence is built from a longest alternating subsequence beginning with a descent and ending with a descent that appears in the first j entries of the permutation— 3421, and then concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n-j$ entries of the permutation— 78956. There are four possibilities for the first piece, namely 32, 42, 31, 41, and there are three possibilities for the second piece, namely, 756, 856, 956. Consider now the permutation 145678923, which satisfies $B_1^{+,9}(145678923)=1$. So $n=9$, $j=1$ and $n-j=8$. The length of the longest alternating subsequence that begins with a descent and ends with an ascent is three. Such an alternating subsequence is built from a longest alternating subsequence beginning with a descent and ending with a descent that appears in the first j entries of the permutation— 1, and then concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n-j$ entries of the permutation— 45678923. There is one possibility for the first piece, namely the null set, and there are six possibilities for the second piece, namely $x23$, with $x\in\{4,5,6,7,8,9\}$. Note that for this to work, it was necessary in (2.6) that a singleton be defined *not* to be an alternating sequence of the type $(-,-)$.

Taking expectations in (3.6), and using (3.5) along with (3.4) and (2.4), we obtain

$$E_n^{\text{sep}}A_n^{+,+}=\frac12c_n^{+,+}+\frac12d_n^{+,+}=\frac{s_1s_{n-1}}{s_n}\big(c_1^{+,-}+\frac12c_{n-1}^{-,+}+\frac12d_{n-1}^{-,+}\big)+$$
$$\sum_{j=2}^{n-1}\frac{\frac12s_js_{n-j}}{s_n}\big(c_j^{+,-}+\frac12c_{n-j}^{-,+}+\frac12d_{n-j}^{-,+}\big)+\frac12c_n^{+,+},n\ge 3,$$

or equivalently (noting from (2.7) that $c_1^{+,-}=1$),

(3.8)
$$\frac{1}{2}d_n^{+,+}=\frac{s_1s_{n-1}}{s_n}\big(1+\frac{1}{2}c_{n-1}^{-,+}+\frac{1}{2}d_{n-1}^{-,+}\big)+\sum_{j=2}^{n-1}\frac{\frac{1}{2}s_js_{n-j}}{s_n}\big(c_j^{+,-}+\frac{1}{2}c_{n-j}^{-,+}+\frac{1}{2}d_{n-j}^{-,+}\big),n\geq 3.$$

Taking expectations in (3.7), and using (3.5) along with (3.4) and (2.4), we obtain

$$E_n^{\text{sep}}A_n^{-,+}=\frac{1}{2}c_n^{-,+}+\frac{1}{2}d_n^{-,+}=\frac{s_1s_{n-1}}{s_n}\big(c_1^{-,-}+\frac{1}{2}c_{n-1}^{-,+}+\frac{1}{2}d_{n-1}^{-,+}\big)+$$
$$\sum_{j=2}^{n-1}\frac{\frac{1}{2}s_js_{n-j}}{s_n}\big(c_j^{-,-}+\frac{1}{2}c_{n-j}^{-,+}+\frac{1}{2}d_{n-j}^{-,+}\big)+\frac{1}{2}c_n^{-,+},n\geq 3,$$

or equivalently (noting from (2.7) that $c_1^{-,-}=0$),

(3.9)
$$\frac{1}{2}d_n^{-,+}=\frac{s_1s_{n-1}}{s_n}\big(\frac{1}{2}c_{n-1}^{-,+}+\frac{1}{2}d_{n-1}^{-,+}\big)+\sum_{j=2}^{n-1}\frac{\frac{1}{2}s_js_{n-j}}{s_n}\big(c_j^{-,-}+\frac{1}{2}c_{n-j}^{-,+}+\frac{1}{2}d_{n-j}^{-,+}\big),n\geq 3.$$

By Lemma 1, we can substitute in equations (3.8) and (3.9) so that each of them is given only in terms of $c_\cdot^{-,-}$ and $c_\cdot^{+,-}$. Indeed, by Lemma 1, we have

$$d_n^{+,+}=c_n^{-,-},\ c_n^{-,+}=c_n^{+,-},d_n^{-,+}=c_n^{+,-}. \tag{3.10}$$

Using (3.10) to substitute in (3.8) and (3.9) and recalling that $s_1=1$, we obtain the two equations

$$\begin{aligned}\frac{1}{2}c_n^{-,-}&=\frac{s_{n-1}}{s_n}\big(1+c_{n-1}^{+,-}\big)+\sum_{j=2}^{n-1}\frac{\frac{1}{2}s_js_{n-j}}{s_n}\big(c_j^{+,-}+c_{n-j}^{+,-}\big),n\geq 3;\\ \frac{1}{2}c_n^{+,-}&=\frac{s_{n-1}}{s_n}c_{n-1}^{+,-}+\sum_{j=2}^{n-1}\frac{\frac{1}{2}s_js_{n-j}}{s_n}\big(c_j^{-,-}+c_{n-j}^{+,-}\big),n\geq 3.\end{aligned} \tag{3.11}$$

Multiplying (3.11) by $2s_nt^n$ and summing over n gives

(3.12)
$$\sum_{n=3}^{\infty}s_nc_n^{-,-}t^n=2t\sum_{n=3}^{\infty}s_{n-1}\big(1+c_{n-1}^{+,-}\big)t^{n-1}+\sum_{n=3}^{\infty}\Big(\sum_{j=2}^{n-1}s_js_{n-j}\big(c_j^{+,-}+c_{n-j}^{+,-}\big)\Big)t^n;$$
$$\sum_{n=3}^{\infty}s_nc_n^{+,-}t^n=2t\sum_{n=3}^{\infty}s_{n-1}c_{n-1}^{+,-}t^{n-1}+\sum_{n=3}^{\infty}\Big(\sum_{j=2}^{n-1}s_js_{n-j}\big(c_j^{-,-}+c_{n-j}^{+,-}\big)\Big)t^n.$$

We have

(3.13)
$$\sum_{n=3}^{\infty}\big(\sum_{j=2}^{n-1}s_js_{n-j}c_j^{+,-}\big)t^n=\sum_{n=3}^{\infty}\big(\sum_{j=1}^{n-1}s_js_{n-j}c_j^{+,-}\big)t^n-t\sum_{n=3}^{\infty}s_{n-1}t^{n-1}=$$
$$\sum_{n=2}^{\infty}\big(\sum_{j=1}^{n-1}s_js_{n-j}c_j^{+,-}\big)t^n-t\sum_{n=2}^{\infty}s_{n-1}t^{n-1}=$$
$$\big(\sum_{n=1}^{\infty}s_nc_n^{+,-}t^n\big)\big(\sum_{n=1}^{\infty}s_nt^n\big)-t\sum_{n=2}^{\infty}s_{n-1}t^{n-1},$$

and similarly,

(3.14)
$$\sum_{n=3}^{\infty}\big(\sum_{j=2}^{n-1}s_js_{n-j}c_{n-j}^{+,-}\big)t^n=\sum_{n=2}^{\infty}\big(\sum_{j=1}^{n-1}s_js_{n-j}c_{n-j}^{+,-}\big)t^n-t\sum_{n=2}^{\infty}s_{n-1}c_{n-1}^{+,-}t^{n-1}=$$
$$\big(\sum_{n=1}^{\infty}s_nt^n\big)\big(\sum_{n=1}^{\infty}s_nc_n^{+,-}t^n\big)-t\sum_{n=2}^{\infty}s_{n-1}c_{n-1}^{+,-}t^{n-1}.$$

Also, noting from (2.7) that $c_1^{-,-}=0$, we have

(3.15)
$$\sum_{n=3}^{\infty}\big(\sum_{j=2}^{n-1}s_js_{n-j}c_j^{-,-}\big)t^n=\sum_{n=2}^{\infty}\big(\sum_{j=1}^{n-1}s_js_{n-j}c_j^{-,-}\big)t^n=\big(\sum_{n=1}^{\infty}s_nc_n^{-,-}t^n\big)\big(\sum_{n=1}^{\infty}s_nt^n\big).$$

Since the conditional probability measure $P_2^{\text{sep}}(\cdot\ |B_1^{+,2}=2)$ gives probability one to the permutation 21, it follows that $c_2^{-,-}=2$. From (2.7), $c_2^{+,-}=1$, $c_1^{+,-}=1$ and $c_1^{-,-}=0$. Using these facts along with the fact that $s_1=1$, $s_2=2$, it follows from (3.12)-(3.15) and (3.2) that

(3.16)
$$G^{-,-}(t)-4t^2=2t(s(t)-t)+2t(G^{+,-}(t)-t)+G^{+,-}(t)s(t)-ts(t)+$$
$$G^{+,-}(t)s(t)-tG^{+,-}(t)$$

and

(3.17)
$$G^{+,-}(t)-2t^2-t=2t(G^{+,-}(t)-t)+G^{-,-}(t)s(t)+G^{+,-}(t)s(t)-tG^{+,-}(t).$$

The equation in (3.16) simplifies to

$$G^{-,-}(t)=\big(2s(t)+t\big)G^{+,-}(t)+ts(t). \tag{3.18}$$

Using (3.18) to substitute for $G^{-,-}(t)$ in (3.17), and performing some algebra, we obtain

$$G^{+,-}(t) = \frac{t(1+s^2(t))}{1-t-s(t)-2s^2(t)-ts(t)}. \tag{3.19}$$

From (2.5), we have

$$s^2(t) = \frac{1}{2}\big(t^2-4t+1-(1-t)\sqrt{t^2-6t+1}\big). \tag{3.20}$$

Using (2.5) and (3.20), we obtain

$$1-t-s(t)-2s^2(t)-ts(t) = -\frac{1}{2}(t^2-6t+1)-\frac{1}{2}(t-3)\sqrt{t^2-6t+1} \tag{3.21}$$

and

$$s^2(t)+1 = \frac{3}{2}-2t+\frac{1}{2}t^2-\frac{1-t}{2}\sqrt{t^2-6t+1}. \tag{3.22}$$

Substituting (3.21) and (3.22) in (3.19), and multiplying the numerator and denominator by -2 yields

$$G^{+,-}(t) = \frac{t\big(-3+4t-t^2+(1-t)\sqrt{t^2-6t+1}\big)}{t^2-6t+1+(t-3)\sqrt{t^2-6t+1}}. \tag{3.23}$$

Writing $-3+4t-t^2 = 1-(t-2)^2 = (1-t+2)(1+t-2) = (t-3)(1-t)$, we have

$$\begin{aligned} &\frac{t\big(-3+4t-t^2+(1-t)\sqrt{t^2-6t+1}\big)}{t^2-6t+1+(t-3)\sqrt{t^2-6t+1}} = \\ &t(1-t)\frac{t-3+\sqrt{t^2-6t+1}}{t^2-6t+1+(t-3)\sqrt{t^2-6t+1}} = \frac{t(1-t)}{\sqrt{t^2-6t+1}}. \end{aligned} \tag{3.24}$$

The formula for $G^{+,-}$ in (3.3) follows from (3.23) and (3.24). The formula for $G^{-,-}$ in (3.3) follows from the formula in (3.3) for $G^{+,-}$, (3.18) and (2.5), along with a little algebra. □

4. Proof of Theorem 1

To prove the theorem, it suffices to prove (1.2) with $E_n^{\text{sep}}A_n^{+,-}$ in place of $E_n^{\text{sep}}A_n$, since $A_n(\sigma)-A_n^{+,-}(\sigma)\in\{0,1,2\}$, for all $\sigma\in S_n$. By Lemma 1, $d^{+,-}=c^{-,+}=c^{+,-}$. From this and (3.4), we have $E_n^{\text{sep}}A_n^{+,-}=c_n^{+,-}$. Thus, from (3.2), it follows that the coefficient of t^n in the power series for $G^{+,-}(t)$ is $s_nE_n^{\text{sep}}A_n^{+,-}$. By Theorem 3 along with Proposition 2, which appears in the appendix, the coefficient of t^n in $G(t)$ is $a_{n-1}-a_{n-2}$, where a_n satisfies (7.28). Thus, we have

(4.1)

$$s_nE_n^{\text{sep}}A_n^{+,-}\sim a_{n-1}-a_{n-2}\sim\frac{1}{2^{\frac54}\sqrt{\pi}n^{\frac12}}\big((3-2\sqrt2)^{-n+\frac12}-(3-2\sqrt2)^{-n+\frac32}\big)=$$
$$\frac{1}{2^{\frac54}\sqrt{\pi}n^{\frac12}}(3-2\sqrt2)^{-n+\frac12}(2\sqrt2-2)=\frac{1}{2^{\frac34}\sqrt{\pi}n^{\frac12}}(3-2\sqrt2)^{-n+\frac12}(2-\sqrt2).$$

By (7.1) in Proposition 1 in the appendix, we have

$$s_n\sim\frac{1}{2^{\frac34}\sqrt{\pi}n^{\frac32}}(3-2\sqrt2)^{-n+\frac12}.$$

Using this with (4.1), we conclude that $E_n^{\text{sep}}A_n^{+,-}\sim(2-\sqrt2)n$, which completes the proof of the theorem. □

5. Generating functions related to the second moment of A_n

Define

(5.1)

$$C_n^{\pm,\pm}=E_n^{\text{sep}}\big((A_n^{\pm,\pm})^2|B_1^{+,n}=n\big);\quad D_n^{\pm,\pm}=E_n^{\text{sep}}\big((A_n^{\pm,\pm})^2|B_1^{-,n}=n\big).$$

Define the generating functions

$$H^{+,-}(t)=\sum_{n=1}^{\infty}s_nC_n^{+,-}t^n;$$
$$H^{-,-}(t)=\sum_{n=1}^{\infty}s_nC_n^{-,-}t^n. \tag{5.2}$$

We will prove the following theorem.

Theorem 4.

(5.3)

$$H^{+,-}(t)=2t^2(1-t)^2\big(t^2-6t+1\big)^{-\frac32}+\frac{-5t^3+8t^2-3t}{t-3}\big(t^2-6t+1\big)^{-\frac12}+$$
$$\frac12t^2(1-t)\big(t^2-6t+1\big)^{\frac12}-\frac12t^2(1-t);$$
$$H^{-,-}(t)=H^{+,-}(t)-\frac{-5t^3+8t^2-3t}{t-3}-\frac12t^2(1-t)^(t^2-6t+1)+$$
$$\frac12t^2(1-t)(t^2-6t+1)^{\frac12}+\frac12t(1-t)-\frac12t(t^2-6t+1)+2t^2(1-t)(t^2-6t+1)^{-\frac12}.$$

Proof. In several equations below, the notation $E_n^{\text{sep}}\big((A_j^{\delta_1,\delta_2}+A_{n-j}^{\delta_3,\delta_4})^2|B_1^{\delta_5,j}=j,B_1^{\delta_6,n-j}=n-j\big)$ is employed, where $\delta_i\in\{+,-\}$, for $i=1,\cdots,6$. This indicates that $A_j^{\delta_1,\delta_2}$ and $A_{n-j}^{\delta_3,\delta_4}$ are independent, with $A_j^{\delta_1,\delta_2}$ considered under the measure $E_j^{\text{sep}}(\cdot\,|B_1^{\delta_5,j}=j)$ and $A_{n-j}^{\delta_3,\delta_4}$ considered under the measure $E_{n-j}^{\text{sep}}(\cdot\,|B_1^{\delta_6,n-j}=n-j)$.

We proceed in the manner of the proof of Theorem 1. Using (3.6), (2.4), (2.2) and (5.1), and noting from (2.7) that $A_1^{+,-}=1$, we have

$$E_n^{\text{sep}}(A_n^{+,+})^2=\frac12C_n^{+,+}+\frac12D_n^{+,+}=$$
$$\frac{s_1s_{n-1}}{s_n}\Big(\frac12E\big((1+A_{n-1}^{-,+})^2|B_1^{+,n-1}=n-1\big)+\frac12E\big((1+A_{n-1}^{-,+})^2|B_1^{-,n-1}=n-1\big)\Big)+$$
$$\sum_{j=2}^{n-1}\frac12\frac{s_js_{n-j}}{s_n}\frac12E\big((A_j^{+,-}+A_{n-j}^{-,+})^2|B_1^{+,j}=j,B_1^{+,n-j}=n-j\big)+$$
$$\sum_{j=2}^{n-1}\frac12\frac{s_js_{n-j}}{s_n}\frac12E\big((A_j^{+,-}+A_{n-j}^{-,+})^2|B_1^{+,j}=j,B_1^{-,n-j}=n-j\big)+\frac12C_n^{+,+},\ n\geq3.$$

Expanding the terms in the equation above, cancelling the term $\frac12C_n^{+,+}$ from both sides and using the notation from (5.1) and (3.1), we have

$$\frac12D_n^{+,+}=\frac{s_1s_{n-1}}{s_n}\big(1+c_{n-1}^{-,+}+\frac12C_{n-1}^{-,+}+d_{n-1}^{-,+}+\frac12D_{n-1}^{-,+}\big)+$$
$$\frac14\sum_{j=2}^{n-1}\frac{s_js_{n-j}}{s_n}\big(C_j^{+,-}+2c_j^{+,-}c_{n-j}^{-,+}+C_{n-j}^{-,+}\big)+\frac14\sum_{j=2}^{n-1}\frac{s_js_{n-j}}{s_n}\big(C_j^{+,-}+2c_j^{+,-}d_{n-j}^{-,+}+D_{n-j}^{-,+}\big),$$
$$n\geq3.$$

By Lemma 1, $D_n^{+,+}=C_n^{-,-},D_n^{-,+}=C_n^{+,-},C_n^{-,+}=C_n^{+,-},c_n^{-,+}=c_n^{+,-},d_n^{-,+}=c_n^{+,-}$, for all n. Making these substitutions in the equation above, multiplying both sides by $2s_n$ and recalling that $s_1=1$ gives

(5.4)
$$s_nC_n^{-,-}=s_{n-1}\big(2+4c_{n-1}^{+,-}+2C_{n-1}^{+,-}\big)+\sum_{j=2}^{n-1}s_js_{n-j}\big(C_j^{+,-}+2c_j^{+,-}c_{n-j}^{+,-}+C_{n-j}^{+,-}\big),\ n\geq3.$$

To derive a second recursion equation similar to the one in (5.4) we consider $A_n^{-,+}$. Using (3.7), (2.4), (2.2) and (5.1) and noting from (2.7) that

$A_1^{-,-}=0$, we have

$$E_n^{\rm sep}(A_n^{-,+})^2=\frac12C_n^{-,+}+\frac12D_n^{-,+}=$$
$$\frac{s_1s_{n-1}}{s_n}\Big(\frac12E\big((A_{n-1}^{-,+})^2|B_1^{+,n-1}=n-1\big)+\frac12E\big((A_{n-1}^{-,+})^2|B_1^{-,n-1}=n-1\big)\Big)+$$
$$\sum_{j=2}^{n-1}\frac12\frac{s_js_{n-j}}{s_n}\frac12E\big((A_j^{-,-}+A_{n-j}^{-,+})^2|B_1^{+,j}=j,B_1^{+,n-j}=n-j\big)+$$
$$\sum_{j=2}^{n-1}\frac12\frac{s_js_{n-j}}{s_n}\frac12E\big((A_j^{-,-}+A_{n-j}^{-,+})^2|B_1^{+,j}=j,B_1^{-,n-j}=n-j\big)+\frac12C_n^{-,+},\ n\ge3.$$

Expanding the terms in the equation above, cancelling the term $\frac12C_n^{-,+}$ from both sides and using the notation from (5.1) and (3.1), we have

$$\frac12D_n^{-,+}=\frac{s_1s_{n-1}}{s_n}\big(\frac12C_{n-1}^{-,+}+\frac12D_{n-1}^{-,+}\big)+\frac14\sum_{j=2}^{n-1}\frac{s_js_{n-j}}{s_n}\big(C_j^{-,-}+2c_j^{-,-}c_{n-j}^{-,+}+C_{n-j}^{-,+}\big)+$$
$$\frac14\sum_{j=2}^{n-1}\frac{s_js_{n-j}}{s_n}\big(C_j^{-,-}+2c_j^{-,-}d_{n-j}^{-,+}+D_{n-j}^{-,+}\big),\ n\ge3.$$

By Lemma 1, $D_n^{-,+}=C_n^{+,-},C_n^{-,+}=C_n^{+,-},c_n^{-,+}=c_n^{+,-},d_n^{-,+}=c_n^{+,-}$, for all n. Making these substitutions in the equation above, multiplying both sides by $2s_n$ and recalling that $s_1=1$ gives

$$s_nC_n^{+,-}=2s_{n-1}C_{n-1}^{+,-}+\sum_{j=2}^{n-1}s_js_{n-j}\big(C_j^{-,-}+2c_j^{-,-}c_{n-j}^{+,-}+C_{n-j}^{+,-}\big),\ n\ge3. \tag{5.5}$$

Multiplying (5.4) and (5.5) by t^n and summing over n gives

(5.6)
$$\sum_{n=3}^\infty s_nC_n^{-,-}t^n=2t\sum_{n=3}^\infty s_{n-1}t^{n-1}+4t\sum_{n=3}^\infty s_{n-1}c_{n-1}^{+,-}t^{n-1}+2t\sum_{n=3}^\infty s_{n-1}C_{n-1}^{+,-}t^{n-1}+$$
$$\sum_{n=3}^\infty\big(\sum_{j=2}^{n-1}s_js_{n-j}C_j^{+,-}\big)t^n+\sum_{n=3}^\infty\big(\sum_{j=2}^{n-1}s_js_{n-j}C_{n-j}^{+,-}\big)t^n+2\sum_{n=3}^\infty\big(\sum_{j=2}^{n-1}s_js_{n-j}c_j^{+,-}c_{n-j}^{+,-}\big)t^n$$

and

$$\begin{aligned}&\sum_{n=3}^\infty s_nC_n^{+,-}t^n=2t\sum_{n=3}^\infty s_{n-1}C_{n-1}^{+,-}t^{n-1}+\sum_{n=3}^\infty\big(\sum_{j=2}^{n-1}s_js_{n-j}C_j^{-,-}\big)t^n+\\&\sum_{n=3}^\infty\big(\sum_{j=2}^{n-1}s_js_{n-j}C_{n-j}^{+,-}\big)t^n+2\sum_{n=3}^\infty\big(\sum_{j=2}^{n-1}s_js_{n-j}c_j^{-,-}c_{n-j}^{+,-}\big)t^n.\end{aligned} \tag{5.7}$$

We have $s_2 = 2$. By (2.7), $C_1^{+,-} = 1$ and $C_1^{-,-} = 0$. By the explanation in the paragraph after (3.15), we have $C_2^{-,-} = 4$ and $C_2^{+,-} = 1$. The first and second double sums on the second line of (5.6) satisfy (3.13) and (3.14) respectively with $c_\cdot^{+,-}$ replaced by $C_\cdot^{+,-}$. The third double sum on the second line of (5.6) satisfies similarly,

$$\sum_{n=3}^{\infty}\big(\sum_{j=2}^{n-1} s_j s_{n-j} c_j^{+,-} c_{n-j}^{+,-}\big)t^n = \big(\sum_{n=1}^{\infty} s_n c_n^{+,-} t^n\big)^2 - t\sum_{n=2}^{\infty} s_{n-1}c_{n-1}^{+,-}t^{n-1}.$$

The double sum on the first line of (5.7) satisfies (3.15) with with $c_\cdot^{-,-}$ replaced by $C_\cdot^{-,-}$ and the first double sum on the second line of (5.7) satisfies (3.14) with $c_\cdot^{+,-}$ replaced by $C_\cdot^{+,-}$. The second double sum on the second line of (5.7) satisfies similarly

$$\sum_{n=3}^{\infty}\big(\sum_{j=2}^{n-1} s_j s_{n-j} c_j^{-,-} c_{n-j}^{+,-}\big)t^n = \big(\sum_{n=1}^{\infty} s_n c_n^{-,-} t^n\big)\big(\sum_{n=1}^{\infty} s_n c_n^{+,-} t^n\big).$$

Using the above facts with (5.2), we obtain from (5.6) and (5.7)

(5.8)
$$H^{-,-}(t) - 8t^2 = 2t(s(t)-t) + 4t\big(G^{+,-}(t)-t\big) + 2t\big(H^{+,-}(t)-t\big) + s(t)H^{+,-}(t) - ts(t) + s(t)H^{+,-}(t) - tH^{+,-}(t) + 2(G^{+,-}(t))^2 - 2tG^{+,-}(t)$$

and

$$H^{+,-}(t) - 2t^2 - t = 2t\big(H^{+,-}(t)-t\big) + s(t)H^{-,-}(t) + s(t)H^{+,-}(t) - tH^{+,-}(t) + 2G^{-,-}(t)G^{+,-}(t). \tag{5.9}$$

The equation in (5.8) simplifies to

$$H^{-,-}(t) = \big(2s(t)+t\big)H^{+,-}(t) + ts(t) + 2tG^{+,-}(t) + 2(G^{+,-}(t))^2. \tag{5.10}$$

From (2.5), $2s(t) + t = 1 - \sqrt{t^2 - 6t + 1}$. Using this, the formula for $H^{-,-}$ in (5.3) follows from (5.10) after substituting for $G^{+,-}$ from (3.3) and performing some algebra.

Using (5.10) to substitute for $H^{-,-}(t)$ in (5.9), and then solving for $H^{+,-}(t)$, one obtains after some algebra,

(5.11)
$$H^{+,-}(t) = \frac{t + ts^2(t) + 2G^{-,-}(t)G^{+,-}(t) + 2ts(t)G^{+,-}(t) + 2s(t)(G^{+,-}(t))^2}{1 - t - s(t) - 2s^2(t) - ts(t)}.$$

Substituting for $G^{+,-}(t)$ and $G^{-,-}(t)$ from (3.3), substituting for $s(t)$ from (2.5) and substituting for $s^2(t)$ from (3.20), and performing a lot of algebra, one finds that the numerator in (5.11) satifies

(5.12)

$$t+ts^2(t)+2G^{-,-}(t)G^{+,-}(t)+2ts(t)G^{+,-}(t)+2s(t)(G^{+,-}(t))^2=$$
$$\frac52t^3-4t^2+\frac32t-\frac{t(1-t)}{2}\sqrt{t^2-6t+1}-\frac{t^2(1-t)^2}{\sqrt{t^2-6t+1}}-\frac{t^2(1-t)^2(t-3)}{t^2-6t+1}.$$

Let

$$X=\sqrt{t^2-6t+1} \tag{5.13}$$

and $Y=t-3$. The denominator of $H^{+,-}(t)$ in (5.11) was calculated in (3.21), and is equal to $-\frac12(X^2+YX)$. We have

(5.14)

$$\frac{1}{X^2+YX}=\frac1Y(\frac1X-\frac{1}{X+Y})=$$
$$\frac1Y(\frac1X+\frac{Y-X}{X^2-Y^2})=\frac{1}{t-3}\Big(\frac{1}{\sqrt{t^2-6t+1}}+\frac{t-3-\sqrt{t^2-6t+1}}{t^2-6t+1-(t-3)^2}\Big)=$$
$$\frac{1}{t-3}\Big(\frac{1}{\sqrt{t^2-6t+1}}-\frac{t-3-\sqrt{t^2-6t+1}}{8}\Big)=$$
$$\frac{1}{(t-3)\sqrt{t^2-6t+1}}-\frac18+\frac{\sqrt{t^2-6t+1}}{8(t-3)}.$$

From (5.11)–(5.14), we obtain

(5.15)

$$H^{+,-}(t)=-2\Big(\frac{1}{(t-3)\sqrt{t^2-6t+1}}-\frac18+\frac{\sqrt{t^2-6t+1}}{8(t-3)}\Big)\times$$
$$\Big((\frac52t^3-4t^2+\frac32t)-\frac{t(1-t)}{2}\sqrt{t^2-6t+1}-\frac{t^2(1-t)^2}{\sqrt{t^2-6t+1}}-\frac{t^2(1-t)^2(t-3)}{t^2-6t+1}\Big).$$

Denote the three expressions in the first parenthetical factor in (5.15) by $\gamma_i, i=1,2,3$, and denote the four expressions in the second parenthetical factor in (5.15) by $\beta_j, j=1,2,3,4$. We multiply out the right hand side of (5.15) in the order of the following double sum: $-2\sum_{j=1}^4\sum_{i=1}^3\gamma_i\beta_j$. We have $\gamma_2\beta_3+\gamma_3\beta_4=0$. Also, after some algebra, one finds that $\big(\gamma_1\beta_3+\gamma_2\beta_4\big)=\frac{t^2(1-t)^2}{8(t-3)}$, which cancels with the term $\gamma_3\beta_3$. Writing down the other seven terms in the order of the double sum above, and recalling the definition of X in (5.13), we obtain

(5.16)

$$H^{+,-}(t)=\frac{-5t^3+8t^2-3t}{t-3}\frac{1}{X}+\frac{5t^3-8t^2+3t}{8}+\frac{-5t^3+8t^2-3t}{8(t-3)}X+$$
$$\frac{t(1-t)}{t-3}-\frac{t(1-t)}{8}X+\frac{t(1-t)}{8(t-3)}(t^2-6t+1)+2t^2(1-t)^2\frac{1}{X^3}.$$

We have

$$\begin{aligned}&\frac{t(1-t)}{t-3}+\frac{t(1-t)}{8(t-3)}(t^2-6t+1)+\frac{5t^3-8t^2+3t}{8}=\\&\frac{1}{8}t(1-t)(t-3)-\frac{1}{8}(-3t+8t^2-5t^3)=-\frac{1}{2}t^2(1-t)\end{aligned}\tag{5.17}$$

and

$$\frac{-5t^3+8t^2-3t}{8(t-3)}-\frac{t(1-t)}{8}=\frac{1}{2}t^2(1-t).\tag{5.18}$$

Now (5.3) for $H^{+,-}(t)$ follows from (5.16)-(5.18). □

6. Proof of Theorem 2

We first proof part (i). Then we make minor additions to that proof to obtain part (ii). By (1.1), it suffices to consider $A_n^{+,-}$ for part (i). By (2.2) and (5.1), we have $E_n^{\text{sep}}(A_n^{+,-})^2=\frac{1}{2}C_n^{+,-}+\frac{1}{2}D_n^{+,-}$, and by Lemma 1, $D_n^{+,-}=C_n^{-,+}=C_n^{+,-}$. Therefore, $E_n^{\text{sep}}(A_n^{+,-})^2=C_n^{+,-}$. Thus $t^n[H^{+,-}]$, the coefficient of t^n in the power series expansion of $H^{+,-}$, is equal to $s_nE_n^{\text{sep}}(A_n^{+,-})^2$. As noted in section 4, $t^n[G^{+,-}]$, the coefficient of t^n in the power series expansion of $G^{+,-}$, is equal to $s_nE_n^{\text{sep}}A_n^{+,-}$. Thus

(6.1)

$$\text{Var}_n^{\text{sep}}(A_n^{+,-})=E_n^{\text{sep}}(A_n^{+,-})^2-\big(E_n^{\text{sep}}A_n^{+,-}\big)^2=\frac{t^n[H^{+,-}]}{s_n}-\big(\frac{t^n[G^{+,-}]}{s_n}\big)^2.$$

Recall that $r_1=3-2\sqrt{2}$ denotes the smaller of the two roots of t^2-6t+1. For some of the calculations below, it will be convenient to write r_1 instead of $3-2\sqrt{2}$ at certain points.

We first consider the asymptotic behavior of $E_n^{\text{sep}}A_n^{+,-}=\frac{t^n[G^{+,-}]}{s_n}$. From 3.3 in Theorem 3 and from Proposition 2 in the appendix, we have

$$t^n[G^{+,-}]=a_{n-1}-a_{n-2},\tag{6.2}$$

where a_n satisfies (7.29). From (7.29),

(6.3)
$$a_{n-i}=\frac{(3-2\sqrt{2})^{-n-\frac12}}{\sqrt{\pi}}\Big(\frac{1}{2^{\frac54}}\frac{(3-2\sqrt{2})^i}{(n-i)^{\frac12}}+\frac{3-4\sqrt{2}}{32\cdot 2^{\frac34}}\frac{(3-2\sqrt{2})^i}{(n-i)^{\frac32}}+o(\frac{1}{n^{\frac32}})\Big), i=1,2.$$

Thus,

$$(6.4)\qquad \begin{aligned} 2^{\frac34}n^{\frac32}a_{n-i}&=\frac{(3-2\sqrt{2})^{-n-\frac12}}{\sqrt{\pi}}\Big(\frac{(3-2\sqrt{2})^i}{\sqrt{2}}n(\frac{n}{n-i})^{\frac12}+\\ &\frac{(3-4\sqrt{2})(3-2\sqrt{2})^i}{32}+o(1)\Big), i=1,2.\end{aligned}$$

Using (6.2) and (6.4), along with (7.2) in the appendix, which gives the asymptotic behavior of s_n, we have

(6.5)
$$\begin{aligned}&E_n^{\rm sep}A_n^{+,-}=\frac{t^n[G^{+,-}]}{s_n}=\frac{1}{3-2\sqrt{2}}\Big(1+\frac{24-9\sqrt{2}}{32}\frac1n+o(\frac1n)\Big)^{-1}\times\\ &\sum_{i=1}^2(-1)^{i-1}\Big(\frac{(3-2\sqrt{2})^i}{\sqrt{2}}n(\frac{n}{n-i})^{\frac12}+\frac{(3-4\sqrt{2})(3-2\sqrt{2})^i}{32}+o(1)\Big)=\\ &\frac{1}{3-2\sqrt{2}}\Big(1-\frac{24-9\sqrt{2}}{32}\frac1n+o(\frac1n)\Big)\times\\ &\sum_{i=1}^2(-1)^{i-1}\Big(\frac{(3-2\sqrt{2})^i}{\sqrt{2}}n(1+\frac{i}{2n})+\frac{(3-4\sqrt{2})(3-2\sqrt{2})^i}{32}+o(1)\Big)=\\ &(2-\sqrt{2})n-\frac14(3-2\sqrt{2})+o(1),\end{aligned}$$

where the last equality follows after some algebra. This proves (1.3). From (6.5) we obtain

$$(6.6)\qquad \big(E_n^{\rm sep}A_n^{+,-}\big)^2=\big(\frac{t^n[G^{+,-}]}{s_n}\big)^2=(6-4\sqrt{2})n^2-\frac{10-7\sqrt{2}}{2}n+o(n).$$

Now we turn to the asymptotic behavior of $E_n^{\rm sep}(A_n^{+,-})^2=\frac{t^n[H^{+,-}]}{s_n}$. Propositions 2 and 3 in the appendix give the asymptotic behavior of the coefficient of t^n in $\big(t^2-6t+1\big)^{-\frac12}$ and in $\big(t^2-6t+1\big)^{-\frac32}$ respectively, and Proposition 1 gives the asymptotic behavior of the coefficient of t^n in $-\frac12\big(t^2-6t+1\big)^{\frac12}$. By these propositions and (5.3) in Theorem 4, which gives the formula for $H^{+,-}(t)$, the leading order contribution to the coefficient of t^n in $H^{+,-}(t)$ comes from the leading order contribution to the coefficient of t^n in the term $2t^2(1-t)^2\big(t^2-6t+1\big)^{-\frac32}$, while the next order contribution to the coefficient of t^n in $H^{+,-}(t)$ comes from the next order contribution to the coefficient of t^n in $2t^2(1-t)^2\big(t^2-6t+1\big)^{-\frac{3}{2}}$ and also from the leading order contribution to the coefficient of t^n in the term $\frac{-5t^3+8t^2-3t}{t-3}\big(t^2-6t+1\big)^{-\frac{1}{2}}$.

We begin with finding the leading order contribution in the coefficient of t^n in $\frac{-5t^3+8t^2-3t}{t-3}\big(t^2-6t+1\big)^{-\frac{1}{2}}$. Recalling from Proposition 2 that a_n denotes the coefficient of t^n in the power series for $(t^2-6t+1)^{-\frac{1}{2}}$, we write

(6.7)
$$\frac{1}{(t-3)\sqrt{t^2-6t+1}}=-\frac{1}{3}\Big(\sum_{n=0}^{\infty}(\frac{t}{3})^n\Big)\Big(\sum_{n=0}^{\infty}a_nt^n\Big)=-\frac{1}{3}\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}(\frac{1}{3})^j a_{n-j}\Big)t^n.$$

We write

$$\sum_{j=0}^{n}(\frac{1}{3})^j a_{n-j}=\sum_{j=0}^{C\log n}(\frac{1}{3})^j a_{n-j}+\sum_{j=C\log n+1}^{n}(\frac{1}{3})^j a_{n-j}, \tag{6.8}$$

where C is chosen so that

$$\sum_{C\log n+1}^{\infty}(\frac{3-2\sqrt{2}}{3})^j=o(n^{-\frac{1}{2}}). \tag{6.9}$$

Considering the asymptotic behavior of a_n in (7.28) in the appendix, we write

$$\begin{aligned}\sum_{j=0}^{C\log n}(\frac{1}{3})^j a_{n-j}=&\frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{2^{\frac{5}{4}}\sqrt{\pi}n^{\frac{1}{2}}}\sum_{j=0}^{C\log n}(\frac{3-2\sqrt{2}}{3})^j+\\&\frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{2^{\frac{5}{4}}\sqrt{\pi}}\sum_{j=0}^{C\log n}(\frac{3-2\sqrt{2}}{3})^j\Big(\frac{1}{(n-j)^{\frac{1}{2}}}-\frac{1}{n^{\frac{1}{2}}}\Big)+\\&\sum_{j=0}^{C\log n}(\frac{1}{3})^j\Big(a_{n-j}-\frac{(3-2\sqrt{2})^{-n+j-\frac{1}{2}}}{2^{\frac{5}{4}}\sqrt{\pi}(n-j)^{\frac{1}{2}}}\Big).\end{aligned} \tag{6.10}$$

From (6.10) and (7.29) in the appendix, and the equality $\sum_{j=0}^{\infty}(\frac{3-2\sqrt{2}}{3})^j=\frac{3}{2\sqrt{2}}$, it follows that

$$\sum_{j=0}^{C\log n}(\frac{1}{3})^j a_{n-j}=\frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{2^{\frac{5}{4}}\sqrt{\pi}n^{\frac{1}{2}}}\Big(\frac{3}{2\sqrt{2}}+o(1)\Big). \tag{6.11}$$

From (7.28), there exits a $C_1 > 0$ such that $a_k \le C_1 \frac{(3-2\sqrt{2})^{-k}}{k^{\frac12}}$, $k \in \mathbb{N}$. Using this with (6.9), we obtain

(6.12)

$$\sum_{j=C\log n+1}^{n} (\frac{1}{3})^j a_{n-j} \le C_1(3-2\sqrt{2})^{-n} \sum_{j=C\log n+1}^{\infty} (\frac{3-2\sqrt{2}}{3})^j = o(\frac{(3-2\sqrt{2})^{-n}}{n^{\frac12}}).$$

From (6.7), (6.8), (6.11) and (6.12), we conclude that the coefficient $-\frac13\sum_{j=0}^n(\frac13)^j a_{n-j}$ of t^n in the power series for $\frac{1}{(t-3)\sqrt{t^2-6t+1}}$ satisfies

$$-\frac13\sum_{j=0}^{n}(\frac13)^j a_{n-j} \sim -\frac13\frac{(3-2\sqrt{2})^{-n-\frac12}}{2^{\frac54}\sqrt{\pi}n^{\frac12}}\frac{3}{2\sqrt{2}} = -\frac{(3-2\sqrt{2})^{-n-\frac12}}{4\cdot 2^{\frac34}\sqrt{\pi}n^{\frac12}}. \tag{6.13}$$

From the above calculations, we conclude that

(6.14)

the leading order contribution in the coefficient of t^n in

$$\frac{-3t+8t^2-5t^3}{t-3}\big(t^2-6t+1\big)^{-\frac12} \text{ is } \frac{(3-2\sqrt{2})^{-n-\frac12}}{4\cdot 2^{\frac34}\sqrt{\pi}}\Big(\frac{3r_1}{(n-1)^{\frac12}}-\frac{8r_1^2}{(n-2)^{\frac12}}+\frac{5r_1^3}{(n-3)^{\frac12}}\Big) =$$

$$\frac{(3-2\sqrt{2})^{-n+\frac12}}{4\cdot 2^{\frac34}\sqrt{\pi}}\Big(\frac{3}{(n-1)^{\frac12}}-\frac{8r_1}{(n-2)^{\frac12}}+\frac{5r_1^2}{(n-3)^{\frac12}}\Big) := I_1.$$

We now consider $2t^2(1-t)^2\big(t^2-6t+1\big)^{-\frac32}$. Writing $2t^2(1-t)^2 = 2t^2 - 4t^3+2t^4$, it follows from Proposition 3 that the two leading orders in the contribution to the coefficient of t^n in $2t^2(1-t)^2\big(t^2-6t+1\big)^{-\frac32}$ are contained in $2\alpha_{n-2}-4\alpha_{n-3}+2\alpha_{n-4}$, where α_n satisfies (7.33). Thus, from (7.33),

(6.15) the leading order contribution to the coefficient of t^n in $2t^2(1-t)^2\big(t^2-6t+1\big)^{-\frac32}$ is given by

$$\frac{(3-2\sqrt{2})^{-n+\frac12}}{4\cdot 2^{\frac34}\sqrt{\pi}}\Big(2(n-2)^{\frac12}-4r_1(n-3)^{\frac12}+2r_1^2(n-4)^{\frac12}\Big) := I_2,$$

and

(6.16)

the second leading order of contribution to the coefficient of t^n in

$2t^2(1-t)^2\big(t^2-6t+1\big)^{-\frac32}$ is given by

$$\frac{(24-9\sqrt{2})(3-2\sqrt{2})^{-n+\frac12}}{128\cdot 2^{\frac34}\sqrt{\pi}}\Big(2(n-2)^{-\frac12}-4r_1(n-3)^{-\frac12}+2r_1^2(n-4)^{-\frac12}\Big) := I_3.$$

By (7.2) in the appendix,

$$\text{the first two leading order terms in } s_n \text{ are} \\ \frac{1}{\sqrt{\pi}}(3-2\sqrt{2})^{-n+\frac{1}{2}}\Big(\frac{1}{2^{\frac{3}{4}}}n^{-\frac{3}{2}} + \frac{12\sqrt{2}-9}{2^{\frac{1}{4}}\cdot 32}n^{-\frac{5}{2}}\Big) := I_4. \tag{6.17}$$

From (6.14)-(6.17), we conclude that

$$\text{the first two leading order terms in } E_n^{\text{sep}}(A_n^{+,-})^2 = \frac{t^n[H^{+,-}]}{s_n} \\ \text{are contained in } \frac{I_1+I_2+I_3}{I_4} \text{ and are of the form } An^2+Bn. \tag{6.18}$$

From (6.14) and (6.17),

$$\frac{I_1}{I_4} = \frac{\frac{1}{4\cdot 2^{\frac{3}{4}}}\big(\frac{3}{(n-1)^{\frac{1}{2}}} - \frac{8r_1}{(n-2)^{\frac{1}{2}}} + \frac{5r_1^2}{(n-3)^{\frac{1}{2}}}\big)}{\frac{1}{2^{\frac{3}{4}}}n^{-\frac{3}{2}} + \frac{12\sqrt{2}-9}{2^{\frac{1}{4}}\cdot 32}n^{-\frac{5}{2}}} = \\ \frac{\frac{n}{4}\big(3(\frac{n}{n-1})^{\frac{1}{2}} - 8r_1(\frac{n}{n-2})^{\frac{1}{2}} + 5r_1^2(\frac{n}{n-3})^{\frac{1}{2}}\big)}{1+\frac{(12\sqrt{2}-9)\sqrt{2}}{32}\frac{1}{n}} = \frac{3-8r_1+5r_1^2}{4}n + O(1) = \\ \frac{3-8(3-2\sqrt{2})+5(3-2\sqrt{2})^2}{4}n + o(n) = (16-11\sqrt{2})n + o(n). \tag{6.19}$$

From (6.15) and (6.17),

$$\frac{I_2}{I_4} = \frac{\frac{1}{4\cdot 2^{\frac{3}{4}}}\big(2(n-2)^{\frac{1}{2}} - 4r_1(n-3)^{\frac{1}{2}} + 2r_1^2(n-4)^{\frac{1}{2}}\big)}{\frac{1}{2^{\frac{3}{4}}}n^{-\frac{3}{2}} + \frac{12\sqrt{2}-9}{2^{\frac{1}{4}}\cdot 32}n^{-\frac{5}{2}}} = \\ \frac{\frac{1}{2}(\frac{n-2}{n})^{\frac{1}{2}} - r_1(\frac{n-3}{n})^{\frac{1}{2}} + \frac{1}{2}r_1^2(\frac{n-4}{n})^{\frac{1}{2}}}{1+\frac{(24-9\sqrt{2})}{32n}}n^2 = \\ \Big(\frac{1}{2}(1-\frac{1}{n}) - r_1(1-\frac{3}{2n}) + \frac{1}{2}r_1^2(1-\frac{2}{n}) + O(\frac{1}{n^2})\Big)\Big(1+\frac{9\sqrt{2}-24}{32n} + O(\frac{1}{n^2})\Big)n^2 = \\ (6-4\sqrt{2})n^2 + (-\frac{79}{4}+\frac{219}{16}\sqrt{2})n + O(1). \tag{6.20}$$

From (6.16) and (6.17),

$$\frac{I_3}{I_4}=\frac{\frac{24-9\sqrt{2}}{128\cdot 2^{\frac{3}{4}}}\big(2(n-2)^{-\frac12}-4r_1(n-3)^{-\frac12}+2r_1^2(n-4)^{-\frac12}\big)}{\frac{1}{2^{\frac34}}n^{-\frac32}+\frac{12\sqrt{2}-9}{32\cdot 2^{\frac14}}n^{-\frac52}}=$$

$$\frac{\frac{24-9\sqrt{2}}{64}\Big((\frac{n}{n-2})^{\frac12}-2r_1(\frac{n}{n-3})^{\frac12}+r_1^2(\frac{n}{n-4})^{\frac12}\Big)}{1+\frac{24-9\sqrt{2}}{32n}}n= \tag{6.21}$$

$$\frac{24-9\sqrt{2}}{64}\big(1-2r_1+r_1^2\big)n+O(1)=\frac{24-9\sqrt{2}}{64}(r_1-1)^2n+O(1)=$$

$$(\frac{24-9\sqrt{2}}{64})(2-2\sqrt{2})^2n+O(1)=\big(\frac{27}{4}-\frac{75}{16}\sqrt{2}\big)n+O(1).$$

From (6.18)-(6.21) we conclude that

$$E_n^{\rm sep}(A_n^{+,-})^2=\frac{t^n[H^{+,-}]}{s_n}= \tag{6.22}$$

$$(6-4\sqrt{2})n^2+\Big(16-11\sqrt{2}-\frac{79}{4}+\frac{219}{16}\sqrt{2}+\frac{27}{4}-\frac{75}{16}\sqrt{2}\Big)n+O(1)=$$

$$(6-4\sqrt{2})n^2+(3-2\sqrt{2})n+O(1).$$

Using (6.22) with (6.6) and (6.1), we conclude that

$$\text{Var}_n^{\rm sep}(A_n^{+,-})=(3-2\sqrt{2}+\frac{10-7\sqrt{2}}{2})n=\frac{16-11\sqrt{2}}{2}n\approx 0.444\,n, \tag{6.23}$$

which proves (1.4).

We now consider part (ii). By (1.1), it suffices to consider $A_n^{-,-}$. We first consider the asymptotic behavior of $E_n^{\rm sep}A_n^{-,-}=\frac{t^n[G^{-,-}]}{s_n}$. From (3.3), we have

$$t^n[G^{-,-}]=t^n[G^{+,-}]-t^n[\frac12 t(t^2-6t+1)^{\frac12}],\ n\ge 3. \tag{6.24}$$

By (7.3) and (7.15) in the appendix, we have

$$t^n[\frac12 t(t^2-6t+1)^{\frac12}]=\frac12 b_{n-1}\sim-\frac{1}{2^{\frac34}\sqrt{\pi}}(3-2\sqrt{2})^{-n+\frac32}\,\frac{1}{n^{\frac32}}. \tag{6.25}$$

From (6.25) and (7.1) in the appendix, we obtain

$$-\frac{t^n[\frac12 t(t^2-6t+1)^{\frac12}]}{s_n}=3-2\sqrt{2}+o(1). \tag{6.26}$$

From (6.24), (6.5) and (6.26), we conclude that

$$E_n^{\rm sep}A_n^{-,-}=\frac{t^n[G^{-,-}]}{s_n}=(2-\sqrt{2})n+\frac34(3-2\sqrt{2})+o(1), \tag{6.27}$$

which proves (1.5).

We now turn to the variance. We have

(6.28)
$$\mathrm{Var}_n^{\mathrm{sep}}(A_n^{-,-}) = E_n^{\mathrm{sep}}(A_n^{-,-})^2 - \big(E_n^{\mathrm{sep}} A_n^{-,-}\big)^2 = \frac{t^n[H^{-,-}]}{s_n} - \Big(\frac{t^n[G^{-,-}]}{s_n}\Big)^2.$$

From (6.27), we have

$$(E_n^{\mathrm{sep}} A_n^{-,-})^2 = \Big(\frac{t^n[G^{-,-}]}{s_n}\Big)^2 = (6-4\sqrt{2})n^2 + \frac{30-21\sqrt{2}}{2}n + o(n). \tag{6.29}$$

From (5.3), we have

(6.30)
$$t^n[H^{-,-}] = t^n[H^{+,-}] + t^n[2t^2(1-t)(t^2-6t+1)^{-\frac{1}{2}}] + \text{ lower order terms.}$$

From Proposition 2 in the appendix, $t^n[2t^2(1-t)(t^2-6t+1)^{-\frac{1}{2}}] = 2(a_{n-2} - a_{n-3})$. Thus, using (7.28) from the appendix, or alternatively, using (4.1) with n replaced by $n-1$, we have

(6.31)
$$t^n[2t^2(1-t)(t^2-6t+1)^{-\frac{1}{2}}] = 2(a_{n-2}-a_{n-3}) \sim \frac{2^{\frac{1}{4}}}{\sqrt{\pi}n^{\frac{1}{2}}}(3-2\sqrt{2})^{-n+\frac{3}{2}}(2-\sqrt{2}).$$

From (6.31) and (7.1), we obtain

$$\frac{t^n[2t^2(1-t)(t^2-6t+1)^{-\frac{1}{2}}]}{s_n} \sim 2(3-2\sqrt{2})(2-\sqrt{2})n = (20-14\sqrt{2})n. \tag{6.32}$$

From (6.30), (6.32) and (6.22), we obtain

$$\begin{aligned} \frac{t^n[H^{-,-}]}{s_n} &= (6-4\sqrt{2})n^2 + (3-2\sqrt{2})n + (20-14\sqrt{2})n + o(n) = \\ &(6-4\sqrt{2})n^2 + (23-16\sqrt{2})n + o(n). \end{aligned} \tag{6.33}$$

From (6.28), (6.29) and (6.33), we conclude that

$$\mathrm{Var}_n^{\mathrm{sep}}(A_n^{-,-}) \sim (23-16\sqrt{2})n - \frac{30-21\sqrt{2}}{2}n = \frac{16-11\sqrt{2}}{2}n,$$

which proves (1.6). □

7. Appendix

Proposition 1.

$$s_n \sim \frac{1}{2^{\frac{3}{4}}\sqrt{\pi}}(3-2\sqrt{2})^{-n+\frac{1}{2}}\, n^{-\frac{3}{2}}. \tag{7.1}$$

With more precision,

$$s_n = \frac{1}{\sqrt{\pi}}(3-2\sqrt{2})^{-n+\frac{1}{2}}\Big(\frac{1}{2^{\frac{3}{4}}}n^{-\frac{3}{2}} + \frac{12\sqrt{2}-9}{2^{\frac{1}{4}}\cdot 32}n^{-\frac{5}{2}} + o(n^{-\frac{5}{2}})\Big). \tag{7.2}$$

Proof. Denote the two roots of t^2-6t+1 by

$$r_1 = 3-2\sqrt{2},\quad r_2 = 3+2\sqrt{2}.$$

Define the sequence $\{b_n\}_{n=0}^\infty$ by

$$\sqrt{t^2-6t+1} = \sum_{n=0}^\infty b_n t^n,\ |t| < 3-2\sqrt{2}. \tag{7.3}$$

Write

$$\sqrt{t^2-6t+1} = \sqrt{1-\frac{t}{r_1}}\sqrt{1-\frac{t}{r_2}}. \tag{7.4}$$

The Taylor series for $\sqrt{1-x}$ is given by

$$\sqrt{1-x} = 1-\sum_{n=1}^\infty \frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}x^n. \tag{7.5}$$

From (7.4) and (7.5), we have

(7.6)

$$\sqrt{t^2-6t+1} = 1-\sum_{n=1}^\infty \frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\frac{1}{r_1^n}t^n - \sum_{n=1}^\infty \frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\frac{1}{r_2^n}t^n +$$
$$\sum_{n=2}^\infty\Big(\sum_{j=1}^{n-1}\frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}}\frac{1}{r_1^{n-j}}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}\frac{1}{r_2^j}\Big)t^n.$$

We write the coefficient of t^n in the second line of (7.6) as

$$\begin{aligned}&\sum_{j=1}^{n-1}\frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}}\frac{1}{r_1^{n-j}}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}\frac{1}{r_2^j} = \\ &\frac{1}{r_1^n}\sum_{j=1}^{n-1}\frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j.\end{aligned} \tag{7.7}$$

From (7.5) we have

(7.8)

$$\sum_{j=1}^\infty \frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j = 1-\sqrt{1-\frac{r_1}{r_2}} = 1-\frac{2^{\frac{5}{4}}}{\sqrt{3+2\sqrt{2}}} = 1-2^{\frac{5}{4}}\sqrt{3-2\sqrt{2}}.$$

From Stirling's formula, $n! \sim n^n e^{-n}\sqrt{2\pi n}$, one obtains the well-known asymptotic formula

$$\frac{(2n)!}{(n!)^2 2^{2n}} \sim \frac{1}{\sqrt{\pi n}}. \tag{7.9}$$

By (7.9), the expression $\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}$ multiplying $(\frac{r_1}{r_2})^j$ in (7.8) is bounded in j. Consequently, for some $C > 0$,

$$\sum_{j=M}^{\infty} \frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j \le C\sum_{j=M}^{\infty}(\frac{r_1}{r_2})^j, \text{for any } M \in \mathbb{N}. \tag{7.10}$$

From this we conclude that

(7.11)

for any $l \in \mathbb{N}$, there exists a constant $C_l > 0$ such that if $M_n = [C_l \log n] \in \mathbb{N}$, then

$$\sum_{j=M_n+1}^{\infty} \frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j \le n^{-l}.$$

By (7.9),

(7.12)

$$\frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}} \sim \frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}, \text{ uniformly over } \ j \in \{1, \cdots, [C_l \log n]\}$$

as $n \to \infty$,

From (7.7), (7.8), (7.11) and (7.12), it follows that the coefficient of t^n in the second line of (7.6) satisfies

$$\begin{aligned}&\sum_{j=1}^{n-1} \frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}}\frac{1}{r_1^{n-j}}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}\frac{1}{r_2^j} \sim \\ &(1-2^{\frac{5}{4}}\sqrt{3-2\sqrt{2}})\frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\frac{1}{r_1^n}.\end{aligned} \tag{7.13}$$

From (7.9) or (7.12), the coefficient of t^n in the first sum on the first line of (7.6) satisfies

$$\frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\frac{1}{r_1^n} \sim \frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\frac{1}{r_1^n}. \tag{7.14}$$

Since $r_2 > r_1$, the coefficient of t^n in the second sum on the first line of (7.6) is exponentially smaller than that of the first sum on the first line of (7.6). Using this last fact with (7.3), (7.6), (7.13) and (7.14), it follows that

$$b_n \sim -2^{\frac{1}{4}}\frac{1}{\sqrt{\pi}}(3-2\sqrt{2})^{-n+\frac{1}{2}}\frac{1}{n^{\frac{3}{2}}}. \tag{7.15}$$

Now (7.1) follows from (7.15), (7.3) and (2.5).

We now turn to (7.2). A refined form of Stirling's formula gives $n! = n^n e^{-n}\sqrt{2\pi n}\big(1+\frac{1}{12n}+O(\frac{1}{n^2})\big)$. Thus, recalling (7.9), we obtain

$$\frac{(2n)!}{(n!)^2 2^{2n}} = \frac{1}{\sqrt{\pi n}}\frac{1+\frac{1}{24n}+O(\frac{1}{n^2})}{\big(1+\frac{1}{12n}+O(\frac{1}{n^2})\big)^2} = \frac{1}{\sqrt{\pi n}}\big(1-\frac{1}{8n}+O(\frac{1}{n^2})\big). \tag{7.16}$$

Thus, the coefficient of t^n in the first sum on the first line of (7.6) satisfies

$$\begin{aligned} &\frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\frac{1}{r_1^n} = \frac{2n}{2n-1}\frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\frac{1}{r_1^n}\big(1-\frac{1}{8n}+O(\frac{1}{n^2})\big) = \\ &\big(1+\frac{1}{2n}+O(\frac{1}{n^2})\big)\frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\frac{1}{r_1^n}\big(1-\frac{1}{8n}+O(\frac{1}{n^2})\big) = \\ &\frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\frac{1}{r_1^n}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big). \end{aligned} \tag{7.17}$$

As before, we don't need to consider the coefficient of t^n in the second sum on the first line of (7.6) since it is exponentially smaller than that of the first sum on the right hand side of (7.6).

Now consider the coefficient of t^n in the second line of (7.6) as given by the right hand side of (7.7). We write this as

$$\begin{aligned} &\frac{1}{r_1^n}\sum_{j=1}^{n-1}\frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j = \\ &\frac{1}{r_1^n}\frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\sum_{j=1}^{n-1}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j + \\ &\frac{1}{r_1^n}\sum_{j=1}^{M_n}\Big(\frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}} - \frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\Big)\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j + \\ &\frac{1}{r_1^n}\sum_{j=M_n}^{n-1}\Big(\frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}} - \frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\Big)\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j := \\ &\frac{1}{r_1^n}\big(I_n + II_n + III_n\big), \end{aligned} \tag{7.18}$$

where $M_n = [C_l \log n]$ with C_l is as in (7.11) and $l > \frac{5}{2}$. Using (7.8), (7.10) and (7.17), we conclude that

$$
\begin{aligned}
I_n &= \big(1 - 2^{\frac{5}{4}}\sqrt{3-2\sqrt{2}} + R_n\big)\frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\big(1 + \frac{3}{8n} + O(\frac{1}{n^2})\big) = \\
&\big(1 - 2^{\frac{5}{4}}\sqrt{3-2\sqrt{2}}\big)\frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\big(1 + \frac{3}{8n} + O(\frac{1}{n^2})\big),
\end{aligned} \tag{7.19}
$$

where R_n decays exponentially.

By the same reasoning that led to (7.10), for some $C > 0$, $|III_n| \leq C\sum_{j=M_n}^{\infty}(\frac{r_1}{r_2})^j$. Thus,

(7.20)

$$
\begin{aligned}
&\text{for any } l \in \mathbb{N}, \text{ there exists a constant } C_l > 0 \text{ such that if we choose} \\
&M_n = [C_l \log n], \text{ then } |III_n| \leq n^{-l}.
\end{aligned}
$$

(Without loss of generality, we can choose the same constant C_l in (7.11) and (7.20).) As above, we choose $l > \frac{5}{2}$.

Consider now II_n. We write

(7.21)

$$
\frac{1}{2(n-j)-1}\frac{(2n-2j)!}{((n-j)!)^2 2^{2(n-j)}} - \frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}} = \frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\big(\frac{B_{n-j}}{B_n} - 1\big),
$$

where $B_k = \frac{1}{2k-1}\frac{(2k)!}{(k!)^2 2^{2k}}$. By (7.17),

(7.22)

$$
\begin{aligned}
&\frac{B_{n-j}}{B_n} = \frac{\frac{1}{2\sqrt{\pi}}\frac{1}{(n-j)^{\frac{3}{2}}}\big(1 + \frac{3}{8(n-j)} + O(\frac{1}{n^2})\big)}{\frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\big(1 + \frac{3}{8n} + O(\frac{1}{n^2})\big)} = \big(\frac{n}{n-j}\big)^{\frac{3}{2}}\big(1 + O(\frac{\log n}{n^2})\big) = \\
&\big(1 + \frac{3}{2}\frac{j}{n-j} + O(\frac{1}{n^2})\big)\big(1 + O(\frac{\log n}{n^2})\big) = 1 + \frac{3}{2}\frac{j}{n} + O(\frac{\log n}{n^2}), \\
&\text{uniformly over } j \in \{1, \cdots, C_l \log n\}.
\end{aligned}
$$

Substituting (7.21) and (7.22) in the formula for II_n in (7.18), we have

$$
II_n = \frac{1}{2n-1}\frac{(2n)!}{(n!)^2 2^{2n}}\sum_{j=1}^{C_l \log n}\big(\frac{3}{2}\frac{j}{n} + O(\frac{\log n}{n^2})\big)\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}\big(\frac{r_1}{r_2}\big)^j. \tag{7.23}
$$

We write

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\big(\frac{3j}{2}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}\big(\frac{r_1}{r_2}\big)^j = \big[\frac{3}{2}x\big(\sum_{j=1}^{\infty}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}x^j\big)\big]'|_{x=\frac{r_1}{r_2}} = \\
&\frac{3}{2}x\big(1 - (1-x)^{\frac{1}{2}}\big)'|_{x=\frac{r_1}{r_2}} = \frac{3}{4}\frac{r_1}{r_2}\big(1 - \frac{r_1}{r_2}\big)^{-\frac{1}{2}} = \frac{3}{8}\frac{r_1^{\frac{3}{2}}}{2^{\frac{1}{4}}},
\end{aligned} \tag{7.24}
$$

where we have used the fact that $r_2 = \frac{1}{r_1}$. Using (7.11) with (7.24), we have

$$\sum_{j=1}^{C_l \log n} \Big(\frac{3}{2}\frac{j}{n}\frac{1}{2j-1}\frac{(2j)!}{(j!)^2 2^{2j}}\Big(\frac{r_1}{r_2}\Big)^j = \frac{1}{n}\Big(\frac{3}{8}\frac{r_1^{\frac{3}{2}}}{2^{\frac{1}{4}}} + O(n^{-l})\Big). \tag{7.25}$$

From (7.17), (7.23) and (7.25), we conclude that

$$\begin{aligned} II_n = \frac{1}{2\sqrt{\pi}}\frac{1}{n^{\frac{3}{2}}}\Big(1+\frac{3}{8n}+O(\frac{1}{n^2})\Big)\Big(\frac{3}{8}\frac{r_1^{\frac{3}{2}}}{2^{\frac{1}{4}}}\frac{1}{n}+O(\frac{\log n}{n^2})\Big) = \\ \frac{3}{16\sqrt{\pi}2^{\frac{1}{4}}}r_1^{\frac{3}{2}}\frac{1}{n^{\frac{5}{2}}}+O(\frac{1}{n^{\frac{7}{2}}}). \end{aligned} \tag{7.26}$$

Now from (7.6), (7.17)-(7.20) and (7.26), we have

$$\begin{aligned} b_n = \frac{(3-2\sqrt{2})^{-n+\frac{1}{2}}}{\sqrt{\pi}}\Big(-\frac{2^{\frac{1}{4}}}{n^{\frac{3}{2}}}-\frac{3}{8}\frac{2^{\frac{1}{4}}}{n^{\frac{5}{2}}}+\frac{3}{16}\frac{3-2\sqrt{2}}{2^{\frac{1}{4}}}\frac{1}{n^{\frac{5}{2}}}+o(\frac{1}{n^{\frac{5}{2}}})\Big) = \\ \frac{(3-2\sqrt{2})^{-n+\frac{1}{2}}}{\sqrt{\pi}}\Big(-\frac{2^{\frac{1}{4}}}{n^{\frac{3}{2}}}+\frac{9-12\sqrt{2}}{16\cdot 2^{\frac{1}{4}}}\frac{1}{n^{\frac{5}{2}}}+o(\frac{1}{n^{\frac{5}{2}}})\Big). \end{aligned} \tag{7.27}$$

Now (7.2) follows from (7.27) and (2.5). □

Proposition 2. *Let $\{a_n\}_{n=0}^\infty$ denote the coefficients of the power series about zero of $\frac{1}{\sqrt{t^2-6t+1}}$: $\frac{1}{\sqrt{t^2-6t+1}} = \sum_{n=0}^\infty a_n t^n$. Then*

$$a_n \sim \frac{1}{2^{\frac{5}{4}}\sqrt{\pi}n^{\frac{1}{2}}}(3-2\sqrt{2})^{-n-\frac{1}{2}}. \tag{7.28}$$

With more precision,

$$a_n = \frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}}\Big(\frac{1}{2^{\frac{5}{4}}}\frac{1}{n^{\frac{1}{2}}}+\frac{3-4\sqrt{2}}{32\cdot 2^{\frac{3}{4}}}\frac{1}{n^{\frac{3}{2}}}+o(\frac{1}{n^{\frac{3}{2}}})\Big). \tag{7.29}$$

Proof. We will show that (7.29) follows readily from (7.33) in Proposition 3 below. Of course (7.28) is contained in (7.29). However, we note that the proof of (7.28) is considerably shorter than the proof of (7.29), just as the proof of (7.1) was considerably shorter than that of (7.2). As was mentioned at the end of section 1, for the proof of Theorem 1 the only results we use from the appendix are (7.1) and (7.28).

Differentiating gives

$$\sum_{n=1}^\infty n a_n t^{n-1} = (3-t)(t^2-6t+1)^{-\frac{3}{2}} = (3-t)\sum_{n=0}^\infty \alpha_n t^n,$$

where α_n is as in Proposition 3. From this we obtain

$$a_n = \frac{3}{n}\alpha_{n-1} - \frac{1}{n}\alpha_{n-2}. \tag{7.30}$$

Thus, from (7.33), we have

(7.31)

$$\alpha_n = \frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}}\Big(\frac{3}{4\cdot 2^{\frac{3}{4}}}\frac{(n-1)^{\frac{1}{2}}}{n} + \frac{3(24-9\sqrt{2})}{128\cdot 2^{\frac{3}{4}}}\frac{1}{(n-1)^{\frac{1}{2}}n}\Big) -$$
$$\frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}}\Big(\frac{3-2\sqrt{2}}{4\cdot 2^{\frac{3}{4}}}\frac{(n-2)^{\frac{1}{2}}}{n} + \frac{(3-2\sqrt{2})(24-9\sqrt{2})}{128\cdot 2^{\frac{3}{4}}}\frac{1}{(n-2)^{\frac{1}{2}}n}\Big) +$$
$$o\Big((3-2\sqrt{2})^{-n}\frac{1}{n^{\frac{3}{2}}}\Big).$$

Writing

$$\frac{(n-i)^{\frac{1}{2}}}{n} = \frac{n^{\frac{1}{2}}(1-\frac{i}{n})^{\frac{1}{2}}}{n} = n^{-\frac{1}{2}} - \frac{i}{2}n^{-\frac{3}{2}} + o(n^{-\frac{3}{2}}),\ i = 1,2,$$

and

$$\frac{1}{(n-i)^{\frac{1}{2}}n} = \frac{1}{n^{\frac{3}{2}}} + o(\frac{1}{n^{\frac{3}{2}}}),\ i = 1,2,$$

and substituting in (7.31), we obtain

(7.32)

$$a_n = \frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}}(\frac{3}{4\cdot 2^{\frac{3}{4}}} - \frac{3-2\sqrt{2}}{4\cdot 2^{\frac{3}{4}}})n^{-\frac{1}{2}} +$$
$$\frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}}(-\frac{1}{2}\frac{3}{4\cdot 2^{\frac{3}{4}}} + \frac{3(24-9\sqrt{2})}{128\cdot 2^{\frac{3}{4}}} + \frac{3-2\sqrt{2}}{4\cdot 2^{\frac{3}{4}}} - \frac{(3-2\sqrt{2})(24-9\sqrt{2})}{128\cdot 2^{\frac{3}{4}}})n^{-\frac{3}{2}} +$$
$$o\Big((3-2\sqrt{2})^{-n}\frac{1}{n^{\frac{3}{2}}}\Big) = \frac{(3-2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}}\Big(\frac{1}{2^{\frac{5}{4}}}\frac{1}{n^{\frac{1}{2}}} + \frac{3-4\sqrt{2}}{32\cdot 2^{\frac{3}{4}}}\frac{1}{n^{\frac{3}{2}}} + o(\frac{1}{n^{\frac{3}{2}}})\Big),$$

which is (7.29).

□

Proposition 3. *Let* $\{\alpha_n\}_{n=0}^{\infty}$ *denote the coefficients of the power series about zero of* $\frac{1}{(t^2-6t+1)^{\frac{3}{2}}}$: $\frac{1}{(t^2-6t+1)^{\frac{3}{2}}} = \sum_{n=0}^{\infty}\alpha_n t^n$. *Then*

$$\alpha_n = \frac{(3-2\sqrt{2})^{-n-\frac{3}{2}}}{\sqrt{\pi}}\Big(\frac{1}{4\cdot 2^{\frac{3}{4}}}n^{\frac{1}{2}} + \frac{24-9\sqrt{2}}{128\cdot 2^{\frac{3}{4}}}\frac{1}{n^{\frac{1}{2}}} + o(\frac{1}{n^{\frac{1}{2}}})\Big). \tag{7.33}$$

Proof. The proof is similar to the proof of (7.2) in Proposition 1. Differentiating (7.5) twice, one obtains

$$(1-x)^{-\frac{3}{2}}=\sum_{n=0}^{\infty}\frac{(2n+1)!}{(n!)^2 2^{2n}}x^n. \tag{7.34}$$

Let r_1, r_2 be the roots of t^2-6t+1 as in the proof of Proposition 1. By (7.4) and (7.34), we have

$$\begin{aligned}\frac{1}{(t^2-6t+1)^{\frac{3}{2}}}=1+\sum_{n=1}^{\infty}\frac{(2n+1)!}{(n!)^2 2^{2n}}\frac{1}{r_1^n}t^n+\sum_{n=1}^{\infty}\frac{(2n+1)!}{(n!)^2 2^{2n}}\frac{1}{r_2^n}t^n+\\ \sum_{n=2}^{\infty}\Big(\sum_{j=1}^{n-1}\frac{(2n-2j+1)!}{((n-j)!)^2 2^{2(n-j)}}\frac{1}{r_1^{n-j}}\frac{(2j+1)!}{(j!)^2 2^{2j}}\frac{1}{r_2^j}\Big)t^n.\end{aligned} \tag{7.35}$$

The coefficient of t^n in the second term on the right hand side of (7.35) is exponentially smaller that of the first term, so we can ignore it. Using (7.16), we have

(7.36)

$$\frac{(2n+1)!}{(n!)^2 2^{2n}}=(2n+1)\frac{1}{\sqrt{\pi n}}\big(1-\frac{1}{8n}+O(\frac{1}{n^2})\big)=(1+\frac{1}{2n})\frac{2n^{\frac{1}{2}}}{\sqrt{\pi}}\big(1-\frac{1}{8n}+O(\frac{1}{n^2})\big)=\frac{2n^{\frac{1}{2}}}{\sqrt{\pi}}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big).$$

So the coefficient of t^n in the first term on the right hand side of (7.35) is given by

$$\frac{2n^{\frac{1}{2}}}{\sqrt{\pi}r_1^n}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big). \tag{7.37}$$

We split the coefficient of t^n in the second line of (7.35) into three parts, I_n, II_n, III_n, just as we did for the coefficient of t^n in the second line of (7.6) in (7.18). Thus, we write this term as

$$
\begin{aligned}
&\sum_{j=1}^{n-1}\frac{(2n-2j+1)!}{((n-j)!)^2 2^{2(n-j)}}\frac{1}{r_1^{n-j}}\frac{(2j+1)!}{(j!)^2 2^{2j}}\frac{1}{r_2^j}=\\
&\frac{1}{r_1^n}\sum_{j=1}^{n-1}\frac{(2n-2j+1)!}{((n-j)!)^2 2^{2(n-j)}}\frac{(2j+1)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j=\\
&\frac{1}{r_1^n}\frac{(2n+1)!}{(n!)^2 2^{2n}}\sum_{j=1}^{n-1}\frac{(2j+1)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j+\\
&\frac{1}{r_1^n}\sum_{j=1}^{M_n}\Big(\frac{(2n-2j+1)!}{((n-j)!)^2 2^{2(n-j)}}-\frac{(2n+1)!}{(n!)^2 2^{2n}}\Big)\frac{(2j+1)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j+\\
&\frac{1}{r_1^n}\sum_{j=M_n}^{n-1}\Big(\frac{(2n-2j+1)!}{((n-j)!)^2 2^{2(n-j)}}-\frac{(2n+1)!}{(n!)^2 2^{2n}}\Big)\frac{(2j+1)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j:=\\
&\frac{1}{r_1^n}\big(I_n+II_n+III_n\big),
\end{aligned}
\tag{7.38}
$$

where $M_n=[C_l\log n]$, with C_l sufficiently large so that

$$
|III_n|\le n^{-l},
\tag{7.39}
$$

similar to (7.20). This time, we choose $l>\frac{1}{2}$.

Similar to the derivation of (7.19), we use (7.34) and (7.36) to obtain

$$
\begin{aligned}
I_n=&\frac{2n^{\frac{1}{2}}}{\sqrt{\pi}}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big)\big((1-\frac{r_1}{r_2})^{-\frac{3}{2}}-1+R_n\big)=\\
&\frac{2n^{\frac{1}{2}}}{\sqrt{\pi}}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big)\big(\frac{1}{8(3-2\sqrt{2})^{\frac{3}{2}}2^{\frac{3}{4}}}-1\big),
\end{aligned}
\tag{7.40}
$$

where R_n decays exponentially.

We now turn to II_n. In the present case we have

$$
II_n=\sum_{j=1}^{M_n}\Big(\frac{(2n-2j+1)!}{((n-j)!)^2 2^{2(n-j)}}-\frac{(2n+1)!}{(n!)^2 2^{2n}}\Big)\frac{(2j+1)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j.
$$

Similar to (7.21) and (7.22), we have

$$
\frac{(2n-2j+1)!}{((n-j)!)^2 2^{2(n-j)}}-\frac{(2n+1)!}{(n!)^2 2^{2n}}=\frac{(2n+1)!}{(n!)^2 2^{2n}}\big(\frac{B_{n-j}}{B_n}-1\big),
$$

where now $B_k = \frac{(2k+1)!}{(k!)^2 2^{2k}}$. By (7.36),

$$\frac{B_{n-j}}{B_n} = \frac{\frac{2(n-j)^{\frac{1}{2}}}{\sqrt{\pi}}\big(1+\frac{3}{8(n-j)}+O(\frac{1}{n^2})\big)}{\frac{2n^{\frac{1}{2}}}{\sqrt{\pi}}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big)} = (\frac{n-j}{n})^{\frac{1}{2}}\big(1+O(\frac{\log n}{n^2})\big) =$$
$$\big(1-\frac{j}{2n}+O(\frac{1}{n^2})\big)\big(1+O(\frac{\log n}{n^2})\big) = 1-\frac{j}{2n}+O(\frac{\log n}{n^2}),$$

uniformly over $j \in \{1, \cdots, C_l \log n\}$.

Thus, similar to (7.23), we obtain

$$II_n = \frac{(2n+1)!}{(n!)^2 2^{2n}}\Big(\sum_{j=1}^{C_l \log n}\big(-\frac{j}{2n}+O(\frac{\log n}{n^2})\big)\frac{(2j+1)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j. \tag{7.41}$$

We write

$$\sum_{j=1}^{\infty}\frac{j}{2}\frac{(2j+1)!}{(j!)^2 2^{2j}}(\frac{r_1}{r_2})^j = \big[\frac{x}{2}\big(\sum_{j=1}^{\infty}\frac{(2j+1)!}{(j!)^2 2^{2j}}x^j\big)'\big]|_{x=\frac{r_1}{r_2}} =$$
$$\big[\frac{x}{2}\big((1-x)^{-\frac{3}{2}}-1\big)'\big]|_{x=\frac{r_1}{r_2}} = \frac{3}{4}\frac{r_1}{r_2}(1-\frac{r_1}{r_2})^{-\frac{5}{2}} = \frac{3}{256\cdot 2^{\frac{1}{4}}(3-2\sqrt{2})^{\frac{1}{2}}}, \tag{7.42}$$

where we have used the fact that $r_2 = \frac{1}{r_1}$. Using (7.42) and (7.36) in (7.41), we obtain, similar to (7.26),

(7.43)

$$II_n = \frac{2n^{\frac{1}{2}}}{\sqrt{\pi}}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big)\big(-\frac{3}{256\cdot 2^{\frac{1}{4}}(3-2\sqrt{2})^{\frac{1}{2}}}\frac{1}{n}+O(\frac{\log n}{n^2})\big) =$$
$$-\frac{1}{\sqrt{\pi}}\frac{3}{128\cdot 2^{\frac{1}{4}}(3-2\sqrt{2})^{\frac{1}{2}}}\frac{1}{n^{\frac{1}{2}}}+o(\frac{1}{n^{\frac{1}{2}}}).$$

From (7.35), (7.37)-(7.40) and (7.43), we obtain

(7.44)

$$\alpha_n = \frac{2n^{\frac{1}{2}}}{\sqrt{\pi}r_1^n}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big)+\frac{2n^{\frac{1}{2}}}{\sqrt{\pi}r_1^n}\big(1+\frac{3}{8n}+O(\frac{1}{n^2})\big)\big(\frac{1}{8(3-2\sqrt{2})^{\frac{3}{2}}2^{\frac{3}{4}}}-1\big)-$$
$$\frac{1}{\sqrt{\pi}}\frac{3}{128\cdot 2^{\frac{1}{4}}(3-2\sqrt{2})^{\frac{1}{2}}}\frac{1}{n^{\frac{1}{2}}} =$$
$$\frac{(3-2\sqrt{2})^{-n-\frac{3}{2}}}{\sqrt{\pi}}\Big(\frac{1}{4\cdot 2^{\frac{3}{4}}}n^{\frac{1}{2}}+\frac{3}{32\cdot 2^{\frac{3}{4}}}\frac{1}{n^{\frac{1}{2}}}-\frac{3(3-2\sqrt{2})}{128\cdot 2^{\frac{1}{4}}}\frac{1}{n^{\frac{1}{2}}}+o(\frac{1}{n^{\frac{1}{2}}})\Big) =$$
$$\frac{(3-2\sqrt{2})^{-n-\frac{3}{2}}}{\sqrt{\pi}}\Big(\frac{1}{4\cdot 2^{\frac{3}{4}}}n^{\frac{1}{2}}+\frac{24-9\sqrt{2}}{128\cdot 2^{\frac{3}{4}}}\frac{1}{n^{\frac{1}{2}}}+o(\frac{1}{n^{\frac{1}{2}}})\Big).$$

□

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Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

Email address: `pinsky@technion.ac.il`

URL: `https://pinsky.net.technion.ac.il/`