EXACT FORMULA AND ASYMPTOTIC BEHAVIOR FOR THE EXPECTED NUMBER OF INVERSIONS IN A RANDOM PERMUTATION AVOIDING A PATTERN OF LENGTH THREE

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ABSTRACT. For $\tau \in S_n$, let $S_n(\tau)$ denote the set of permutations in S_n which avoid the pattern τ, and let E_n^τ denote the expectation with respect to the uniformly random probability measure on $S_n(\tau)$. Let $\mathcal{I}_n(\sigma)$ denote the number of inversions in $\sigma \in S_n$. We study $E_n^\tau \mathcal{I}_n$ for $\tau \in \{231, 132, 213, 312\} \subset S_3$. We prove that

$$E_n^{231}\mathcal{I}_n = E_n^{312}\mathcal{I}_n = \frac{1}{2}\frac{n!(n+1)!4^n}{(2n)!} - \frac{1}{2}(3n+1),$$

and that

$$E_n^{132}\mathcal{I}_n = E_n^{213}\mathcal{I}_n = \frac{1}{2}(n-1)n - E_n^{231}\mathcal{I}_n.$$

From the first equation it follows that

$$E_n^{231}\mathcal{I}_n = E_n^{312}\mathcal{I}_n \sim \frac{\sqrt{\pi}}{2}n^{\frac{3}{2}}.$$

We also show that the variance $\mathrm{Var}_{P_n^\tau}(\mathcal{I}_n)$ of $\mathcal{I}_n$ under P_n^τ satisfies

$$\mathrm{Var}_{P_n^\tau}(\mathcal{I}_n) \sim (\frac{5}{6} - \frac{\pi}{4})n^3 \approx 0.048n^3, \text{ for } \tau \in \{231, 132, 213, 312\}.$$

1. Introduction and Statement of Results

Recall the definition of pattern avoidance for permutations. Let S_n denote the set of permutations of $[n] := \{1, \cdots, n\}$. If $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ and $\tau = \tau_1\cdots\tau_m \in S_m$, where $2 \leq m \leq n$, then we say that σ contains τ as a pattern if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that for all $1 \leq j, k \leq m$, the inequality $\sigma_{i_j} < \sigma_{i_k}$ holds if and only if the inequality $\tau_j < \tau_k$ holds. If σ does not contain τ, then we say that σ *avoids* τ. Denote by $S_n(\tau)$ the set of permutation in S_n that avoid τ. (If $\tau \in S_m$ and $m > n$, then we say that every $\sigma \in S_n$ avoids τ.)

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In this paper we obtain an explicit formula and give the asymptotic behavior for the expected number of inversions in a permutation chosen uniformly at random from $S(\tau)$, where $\tau \in S_3$ satisfies $\tau \in \{231, 132, 213, 312\}$. Recall that the number of inversions $\mathcal{I}_n(\sigma)$ in a permutation $\sigma \in S_n$ is given by

$$\mathcal{I}_n(\sigma) = \sum_{1 \le i < j \le n} 1_{\sigma_j < \sigma_i}.$$

As $\sigma \in S_n$ varies, $\mathcal{I}_n(\sigma)$ takes on all integral values between 0 and $\frac{1}{2}(n-1)n$. By symmetry, the expected number of inversions in a uniformly random permutation from S_n is $\frac{1}{4}(n-1)n$.

Let P_n^τ denote the uniform distribution on $S_n(\tau)$ and let E_n^τ denote the corresponding expectation. It is well-known that $|S_n(\tau)| = C_n$, for all six permutations $\tau \in S_3$, where $C_n = \frac{\binom{2n}{n}}{n+1}$ is the nth Catalan number [3, 10].

Theorem 1.

$$E_n^{231}\mathcal{I}_n = E_n^{312}\mathcal{I}_n = \frac{1}{2}\frac{4^n}{C_n} - \frac{1}{2}(3n+1) = \frac{1}{2}\frac{n!(n+1)!4^n}{(2n)!} - \frac{1}{2}(3n+1), \tag{1.1}$$

and

$$E_n^{231}\mathcal{I}_n = E_n^{312}\mathcal{I}_n \sim \frac{\sqrt{\pi}}{2} n^{\frac{3}{2}}. \tag{1.2}$$

Also,

$$E_n^{132}\mathcal{I}_n = E_n^{213}\mathcal{I}_n = \frac{1}{2}(n-1)n - E_n^{231}\mathcal{I}_n. \tag{1.3}$$

Remark. One can easily check that the total number of inversions in all of the $C_4 = 14$ permutations in $S_4(231)$ is equal to 37, and with a little more effort, one can check that the total number of inversions in all of the $C_5 = 42$ permutations in $S_5(231)$ is equal to 176. Thus, $E_4^{231}\mathcal{I}_4 = \frac{37}{14}$ and $E_5^{231}\mathcal{I}_5 = \frac{88}{21}$. The reader can check that the right hand side of (1.1) reproduces this for $n = 4, 5$.

It follows from the theorem that under P_n^τ, the expected value of the fraction of pairs $\{(\sigma_i, \sigma_j)\}_{1 \le i < j \le n}$ that are inverted converges to 0 or 1 respectively as $n \to \infty$, according to whether τ itself has two inversions or one inversion respectively. This expected value also converges to 0 if $\tau = 321$, and to 1 if $\tau = 123$. Indeed, a look at the Simion-Schmidt bijection [3, 10] between $S_n(132)$ and $S_n(123)$, or equivalently, between $S_n(231)$ and $S_n(321)$,

easily reveals that $\mathcal{I}_n$ under P_n^{123} stochastically dominates $\mathcal{I}_n$ under P_n^{132}, and that $\mathcal{I}_n$ under P_n^{231} stochastically dominates $\mathcal{I}_n$ under P_n^{321}.

Since $\mathcal{I}_n(\sigma) \leq \frac{1}{2}(n-1)n$, for all $\sigma \in S_n$, it follows trivially from Theorem 1 that the weak law of large numbers holds for $\mathcal{I}_n$ under P_n^τ, for $\tau \in \{132, 213\}$:

$$\mathrm{w} - \lim_{n\to\infty} \frac{\mathcal{I}_n}{E_n^\tau \mathcal{I}_n} = 1, \text{ for } \tau \in \{132, 213\}.$$

To study the concentration of $\mathcal{I}_n$ under P_n^τ for $\tau \in \{231, 312\}$, we now consider the variance $\mathrm{Var}_{P_n^\tau}(\mathcal{I}_n)$ of $\mathcal{I}_n$.

Theorem 2.

$$Var_{P_n^\tau}(\mathcal{I}_n) \sim (\frac{5}{6} - \frac{\pi}{4})n^3 \approx 0.048n^3, \text{ for } \tau \in \{231, 132, 213, 312\}. \tag{1.4}$$

The following corollary of Theorems 1 and 2 is an immediate application of Chebyshev's inequality.

Corollary 1.

$$\limsup_{n\to\infty} P_n^\tau(|\frac{\mathcal{I}_n}{\frac{\sqrt{\pi}}{2}n^{\frac{3}{2}}} - 1| > a) \leq \frac{\frac{5}{6} - \frac{\pi}{4}}{a^2} \approx \frac{0.048}{a^2}, \text{ for } \tau \in \{231, 312\}. \tag{1.5}$$

Remark. Letting $a = 1$ in the corollary gives

$$\limsup_{n\to\infty} P_n^\tau(\mathcal{I}_n \geq \sqrt{\pi}n^{\frac{3}{2}}) \leq \frac{5}{6} - \frac{\pi}{4} \approx 0.048, \text{for } \tau \in \{231, 312\}$$

while letting $a = \frac{1}{2}$ gives

$$\limsup_{n\to\infty} P_n^\tau(\frac{1}{4}\sqrt{\pi}n^3 \leq \mathcal{I}_n \leq \frac{3}{4}\sqrt{\pi}n^{\frac{3}{2}}) \geq 1-(\frac{10}{3}-\pi) \approx 0.808, \text{for } \tau \in \{231, 312\}.$$

It is possible, of course, that the weak law of large number holds for $\mathcal{I}_n$ under P_n^τ, for $\tau \in \{231, 312\}$, even though the second moment method fails.

The perspective taken in this paper is in some sense the obverse of the perspective taken in [8]. In that paper, the asymptotic probability as $n \to \infty$ of $S_n(\tau)$, for $\tau \in S_3$, was studied under Mallows distributions. The Mallows distribution P_n^q on S_n, where $q \in (0, \infty)$, is defined by $P_n^q(\sigma) = \frac{q^{\mathcal{I}_n(\sigma)}}{N_n(q)}$, where $N_n(q)$ is the appropriate normalization constant. Thus, P_n^q favors permutations with few inversions if $q \in (0, 1)$ and favors permutations with many inversions if $q > 1$.

We are unaware of any articles in the literature concerning the explicit enumeration of the inversion statistic for pattern avoiding permutations.

The paper [4] investigates the analog of Wilf equivalence for the inversion statistic in pattern-avoiding permutations: two patterns τ_1 and τ_2 are equivalent with respect to the inversion statistic if the distribution of the inversion statistic of a uniformly distributed permutation from $S_n(\tau_1)$ and from $S_n(\tau_2)$ coincide for all n. Enumeration of fixed points for pattern avoiding permutations has been considered in [5, 6, 9]. Enumeration of other statistics, including ascents, descents, double ascents, double descents, peaks and valleys can be found among the papers [1, 2, 7].

We end this introduction by noting that it suffices to prove Theorems 1 and 2 for just one of the four choices of τ. To see this, recall that the reverse σ^{rev} of a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is the permutation $\sigma^{\text{rev}} := \sigma_n \cdots \sigma_1$, and the complement σ^{comp} of σ is the permutation satisfying $\sigma_j^{\text{comp}} = n + 1 - \sigma_j$, $j \in [n]$. Let $\sigma^{\text{rev-comp}}$ denote the reverse of the complement of σ, or equivalently, the complement of the reverse of σ. Clearly, $\sigma \in S_n(\tau)$ if and only if $\sigma^* \in S_n(\tau^*)$, for $*$ equal to any of the three transformations on permutations that we just defined. Also, one has the identities

$$\begin{aligned}
&\mathcal{I}_n(\sigma) + \mathcal{I}_n(\sigma^{\text{rev}}) = \frac{1}{2}(n-1)n, \ \sigma \in S_n; \\
&\mathcal{I}_n(\sigma) + \mathcal{I}_n(\sigma^{\text{comp}}) = \frac{1}{2}(n-1)n, \ \sigma \in S_n; \\
&\mathcal{I}_n(\sigma) = \mathcal{I}_n(\sigma^{\text{rev-comp}}), \ \sigma \in S_n.
\end{aligned}$$

Thus, from the first two identities above, if we prove Theorem 1 for say $\tau = 231$, then the theorem also holds for $\tau = 132 = 231^{\text{rev}}$ and for $\tau = 213 = 231^{\text{comp}}$. And then since the theorem now holds for 213, it also holds for $\tau = 312 = 213^{\text{rev}}$. If we prove Theorem 2 for say $\tau = 231$, then from the three identities above, it also holds for $\tau \in \{132, 213, 312\}$.

Using generating function techniques, we prove Theorem 1 in section 2 and Theorem 2 in section 3.

2. Proof of Theorem 1

As shown in the last paragraph of the first section, it suffices to prove the theorem for $\tau = 231$. If $\sigma \in S_n(231)$ satisfies $\sigma_j = n$, then necessarily $\{\sigma_1, \cdots, \sigma_{j-1}\} = \{1, \cdots, j-1\}$ and $\{\sigma_{j+1}, \cdots, \sigma_n\} = \{j+1, \cdots, n\}$.

Furthermore the permutation $\sigma_1 \cdots \sigma_{j-1}$ belongs to $S_{j-1}(231)$ and the permutation $\sigma' \in S_{n-j}$ obtained by the relative order of $\sigma_{j+1} \cdots \sigma_n$ satisfies $\sigma' \in S_{n-j}(231)$. This correspondence is of course reversible. From this it follows that

$$P_n^{231}(\sigma_j = n) = \frac{C_{j-1}C_{n-j}}{C_n},\ j = 1, \cdots, n;\ n \in \mathbb{N}, \tag{2.1}$$

and that

$$E_n^{231}(\mathcal{I}_n | \sigma_j = n) = E_{j-1}^{231}\mathcal{I}_{j-1} + E_{n-j}^{231}\mathcal{I}_{n-j} + (n-j). \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$\begin{aligned} E_n^{231}\mathcal{I}_n = \sum_{j=1}^{n} \left(E_{j-1}^{231}\mathcal{I}_{j-1} + E_{n-j}^{231}\mathcal{I}_{n-j} + (n-j)\right)\frac{C_{j-1}C_{n-j}}{C_n} = \\ n - \sum_{j=1}^{n} j\frac{C_{j-1}C_{n-j}}{C_n} + 2\sum_{j=1}^{n} E_{j-1}^{231}\mathcal{I}_{j-1}\frac{C_{j-1}C_{n-j}}{C_n}. \end{aligned} \tag{2.3}$$

Letting

$$d_n = E_n^{231}\mathcal{I}_n, \tag{2.4}$$

and letting $l = j - 1$, we can rewrite (2.3) as

$$C_n d_n = nC_n - \sum_{l=0}^{n-1} lC_l C_{n-l-1} - \sum_{l=0}^{n-1} C_l C_{n-l-1} + 2\sum_{l=0}^{n-1} d_l C_l C_{n-l-1}. \tag{2.5}$$

Let

$$C(t) = \sum_{j=0}^{\infty} C_n t^n \tag{2.6}$$

denote the generating function for the Catalan numbers. As is well known,

$$C(t) = \frac{1 - (1-4t)^{\frac{1}{2}}}{2t}. \tag{2.7}$$

Note that

$$tC'(t) = \sum_{n=0}^{\infty} nC_n t^n. \tag{2.8}$$

Define

$$D(t) = \sum_{n=0}^{\infty} C_n d_n t^n. \tag{2.9}$$

Multiplying both sides of (2.5) by t^n and summing over n from 0 to ∞, we obtain

(2.10)
$$D(t) = tC'(t) - t\sum_{n=0}^{\infty}\sum_{l=0}^{n-1} lC_l t^l C_{n-l-1}t^{n-l-1} - t\sum_{n=0}^{\infty}\sum_{l=0}^{n-1} C_l t^l C_{n-l-1}t^{n-l-1}$$
$$+ 2t\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} C_l d_l t^l C_{n-l-1}t^{n-l-1}.$$

From (2.6) and (2.8)–(2.10), it follows that

$$D(t) = tC'(t) - t^2C(t)C'(t) - tC^2(t) + 2tD(t)C(t), \tag{2.11}$$

from which we conclude that

$$D(t) = \frac{tC'(t) - t^2C(t)C'(t) - tC^2(t)}{1 - 2tC(t)}. \tag{2.12}$$

The denominator of (2.12) satisfies

$$1 - 2tC(t) = (1-4t)^{\frac{1}{2}}. \tag{2.13}$$

We have

$$C'(t) = -\frac{1}{2t^2} + \frac{(1-4t)^{\frac{1}{2}}}{2t^2} + \frac{(1-4t)^{-\frac{1}{2}}}{t}. \tag{2.14}$$

Thus,

$$tC'(t) + C(t) = (1-4t)^{-\frac{1}{2}}. \tag{2.15}$$

From (2.7), (2.14) and (2.15), we can write the numerator of (2.12) as

$$tC'(t) - t^2C(t)C'(t) - tC^2(t) = tC'(t) - tC(t)\big(tC'(t) + C(t)\big) =$$
$$-\frac{1}{2t} + \frac{(1-4t)^{\frac{1}{2}}}{2t} + (1-4t)^{-\frac{1}{2}} - \frac{1-(1-4t)^{\frac{1}{2}}}{2}(1-4t)^{-\frac{1}{2}} = \tag{2.16}$$
$$-\frac{1}{2t} + \frac{(1-4t)^{\frac{1}{2}}}{2t} + \frac{(1-4t)^{-\frac{1}{2}}}{2} + \frac{1}{2}$$

From (2.12), (2.13) and (2.16), we conclude that

$$D(t) = \frac{1}{2}\Big(-\frac{(1-4t)^{-\frac{1}{2}}}{t} + \frac{1}{t} + (1-4t)^{-1} + (1-4t)^{-\frac{1}{2}}\Big). \tag{2.17}$$

The Taylor series for $(1-4t)^{-\frac{1}{2}}$ is given by

$$(1-4t)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{2n}{n}t^n, \tag{2.18}$$

and of course

$$(1-4t)^{-1} = \sum_{n=0}^{\infty} 4^n t^n. \tag{2.19}$$

From (2.17),(2.18) and (2.19), we conclude that the coefficient $C_n d_n$ of t^n in the power series representation of $D(t)$ satisfies

(2.20)
$$2C_n d_n = -\binom{2n+2}{n+1} + 4^n + \binom{2n}{n} = 4^n - \binom{2n}{n}\frac{3n+1}{n+1} = 4^n - (3n+1)C_n.$$

Thus,

$$E_n^{231}\mathcal{I}_n = d_n = \frac{1}{2}\frac{4^n}{C_n} - \frac{1}{2}(3n+1),$$

which is (1.1). Using Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$ with (1.1) readily yields (1.2). □

3. Proof of Theorem 2

As shown in the last paragraph of the first section, it suffices to prove the theorem for $\tau = 231$. The same reasoning as in the first paragraph of section 2 gives

$$E_n^{231}(\mathcal{I}_n^2 | \sigma_j = n) = E\big(n - j + X + Y\big)^2, \tag{3.1}$$

where X is distributed as $\mathcal{I}_{j-1}$ under P_{j-1}^{231}, Y is distributed as $\mathcal{I}_{n-j}$ under P_{n-j}^{231}, and X and Y are independent. Let

$$g_n = E_n^{231}\mathcal{I}_n^2. \tag{3.2}$$

Using (3.1) and (3.2) with (2.1), and recalling the definition of d_n in (2.4), we have

(3.3)
$$g_n = E_n^{231}\mathcal{I}_n^2 = \sum_{j=1}^{n} g_{j-1}\frac{C_{j-1}C_{n-j}}{C_n} + \sum_{j=1}^{n} g_{n-j}\frac{C_{j-1}C_{n-j}}{C_n} + \sum_{j=1}^{n}(n-j)^2\frac{C_{j-1}C_{n-j}}{C_n} +$$
$$2\sum_{j=1}^{n}(n-j)(d_{j-1}+d_{n-j})\frac{C_{j-1}C_{n-j}}{C_n} + 2\sum_{j=1}^{n} d_{j-1}d_{n-j}\frac{C_{j-1}C_{n-j}}{C_n}.$$

Multiplying on both sides by C_n and letting $l = j-1$, we rewrite (3.3) as

(3.4)
$$C_n g_n = \sum_{l=0}^{n-1} g_l C_l C_{n-l-1} + \sum_{l=0}^{n-1} g_{n-l-1} C_l C_{n-l-1} + \sum_{l=0}^{n-1} (n-l-1)^2 C_l C_{n-l-1} +$$
$$2\sum_{l=0}^{n-1} (n-l-1)(d_l + d_{n-l-1}) C_l C_{n-l-1} + 2\sum_{l=0}^{n-1} d_l d_{n-l-1} C_l C_{n-l-1}.$$

Let

$$G(t) = \sum_{n=0}^{\infty} C_n g_n t^n, \tag{3.5}$$

Multiplying both sides of (3.4) by t^n and summing over n from 0 to ∞, and noting that the first two terms on the right hand side of (3.4) are identical, we obtain

(3.6)
$$G(t) = 2t\sum_{n=0}^{\infty}\sum_{l=0}^{n-1} C_l g_l t^l C_{n-l-1} t^{n-l-1} + t\sum_{n=0}^{\infty}\sum_{l=0}^{n-1} C_l t^l (n-l-1)^2 C_{n-l-1} t^{n-l-1} +$$
$$2t\sum_{n=0}^{\infty}\sum_{l=0}^{n-1} C_l d_l t^l (n-l-1) C_{n-l-1} t^{n-l-1} + 2t\sum_{n=0}^{\infty}\sum_{l=0}^{n-1} C_l t^l (n-l-1) C_{n-l-1} d_{n-l-1} t^{n-l-1} +$$
$$2t\sum_{n=0}^{\infty}\sum_{l=0}^{n-1} C_l d_l t^l C_{n-l-1} d_{n-l-1} t^{n-l-1}.$$

From (2.6), we have

$$t^2 C''(t) + tC'(t) = \sum_{n=0}^{\infty} n^2 C_n t^n. \tag{3.7}$$

From (2.9) we have

$$tD'(t) = \sum_{n=0}^{\infty} n C_n d_n t^n. \tag{3.8}$$

From (3.5)–(3.8) along with (2.6), (2.8) and (2.9), it follows that

$$G(t) = 2tG(t)C(t) + tC(t)\big(t^2 C''(t) + tC'(t)\big) + 2t\big(D(t)tC'(t)\big) + 2t\big(C(t)tD'(t)\big) + 2tD^2(t),$$

from which we conclude that

(3.9)

$$G(t) = \frac{t}{1-2tC(t)}\big(t^2C(t)C''(t) + tC(t)C'(t) + 2tD(t)C'(t) + 2tC(t)D'(t) + 2D^2(t)\big) =$$

$$t(1-4t)^{-\frac{1}{2}}\big(t^2C(t)C''(t) + tC(t)C'(t) + 2tD(t)C'(t) + 2tC(t)D'(t) + 2D^2(t)\big),$$

where the last inequality follows from (2.7).

From (2.7), (2.17) and (3.9), it follows that the right hand side of (3.9), and consequently also the function $G(t)$, is the finite sum of terms of the form $at^b(1-4t)^{-c}$, where $a \in \mathbb{R}, b \in \mathbb{Z}$ and c satisfies either $c \in \mathbb{Z}^+$ or $c \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$. Let $[(1-4t)^{-c}]_n$ denote the coefficient of t^n in the power series expansion of $(1-4t)^{-c}$. From (2.7) and Stirling's formula, we have $[(1-4t)^{-\frac{1}{2}}]_n = \binom{2n}{n} = 4^n\theta(n^{-\frac{1}{2}})$. Since $\big((1-4t)^l\big)' = -4l(1-4t)^{l-1}$, it is immediate that

$$[(1-4t)^{-\frac{1}{2}-l}]_n = 4^n\theta(n^{-\frac{1}{2}+l}),\ l \in \mathbb{Z}. \tag{3.10}$$

The same considerations applied to (2.19) show that

$$[(1-4t)^{-l}]_n = 4^n\theta(n^{l-1}),\ l \in \mathbb{N}. \tag{3.11}$$

In light of the above paragraph and (3.9), the order of the coefficient C_ng_n of t^n in the power series expansion of $G(t)$ will be 4^nt^{c-1}, where $-c$ is the smallest exponent of $(1-4t)$ appearing among the various terms of the form $at^b(1-4t)^{-c}$ that comprise the right hand side of (3.9) (unless, there is more than one term with this smallest exponent and these terms cancel each other out, however we shall see that this does not happen).

From (2.7), we see that $C(t) = \frac{1}{2}t^{-1} - \frac{1}{2}t^{-1}(1-4t)^{\frac{1}{2}}$ is the sum of two terms of the form $at^b(1-4t)^{-c}$, one of which has $-c = 0$ and the other of which has $-c = \frac{1}{2}$. Term by term differentiation of $C(t)$ shows that $C'(t)$ and $C''(t)$ are each the sum of terms of the above noted form with the smallest exponent $-c$ being $-c = -\frac{1}{2}$ in the case of $C'(t)$ and $-c = -\frac{3}{2}$ in the case of $C''(t)$. From (2.17), we see that in the representation of $D(t)$ as the sum of terms of the above noted form, the smallest exponent $-c$ is -1, and thus in the representation for $D'(t)$, the smallest exponent $-c$ is -2.

Consider now the five summands between the parentheses on the right hand of (3.9). The first term is $t^2C(t)C''(t)$. The smallest exponent $-c$ in

the representation for $t^2C(t)C''(t)$ is $-\frac{3}{2}$, obtained by adding the smallest exponent $-c = 0$ in the representation of $C(t)$ to the smallest exponent $-c = -\frac{3}{2}$ in the representation of $C''(t)$. Similarly the smallest exponents $-c$ in the representations of the other four terms are as follows: for $tC(t)C'(t)$ it is $-c = 0 + (-\frac{1}{2}) = -\frac{1}{2}$; for $2D(t)C'(t)$ it is $-c = -1 + (-\frac{1}{2}) = -\frac{3}{2}$; for $2tC(t)D'(t)$ is is $-c = 0 + (-2) = -2$; for $2D^2(t)$ it is $-c = (-1) + (-1) = -2$. Thus, the smallest exponent $-c$ in the sum of the five terms between the parentheses on the right hand side of (3.9) is $c = -2$, with a contribution coming from the term $2tC(t)D'(t)$ and a contribution coming from $2D^2(t)$. Differentiating (2.17), we find that the term in $D'(t)$ with the exponent $-c = -2$ is $2(1-4t)^{-2}$. The term in $C(t)$ with exponent $-c = 0$ is $\frac{1}{2}t^{-1}$. Thus, the term in $2tC(t)D'(t)$ with the exponent $c = -2$ is $2t(\frac{1}{2}t^{-1})(2(1-4t)^{-2}) = 2(1-4t)^{-2}$. From (2.17), the term in $D(t)$ with the exponent $-c = -1$ is $\frac{1}{2}(1-4t)^{-1}$. Thus, the term in $2D^2(t)$ with the exponent $c = -2$ is $2(\frac{1}{2}(1-4t)^{-1})^2 = \frac{1}{2}(1-4t)^{-2}$.

Recall that the five summands on the right hand side of (3.9) are multiplied by $t(1-4t)^{-\frac{1}{2}}$. Thus, it follows from the above analysis that the leading order asymptotic behavior of the term C_ng_n in the power series expansion of $G(t)$ is equal to the leading order asymptotic behavior of the coefficient of t^n in the power series expansion of $t(1-4t)^{-\frac{1}{2}}\big(2(1-4t)^{-2} + \frac{1}{2}(1-4t)^{-2}\big) = \frac{5}{2}t(1-4t)^{-\frac{5}{2}}$. Differentiating (2.18) twice shows that

$$(1-4t)^{-\frac{5}{2}} = \frac{1}{12}\sum_{n=2}^{\infty} n(n-1)\binom{2n}{n}t^{n-2}.$$

Thus,

$$\frac{5}{2}t(1-4t)^{-\frac{5}{2}} = \frac{5}{24}\sum_{n=2}^{\infty} n(n-1)\binom{2n}{n}t^{n-1} = \frac{5}{24}\sum_{n=1}^{\infty}(n+1)n\binom{2n+2}{n+1}t^n.$$

We conclude that

$$C_ng_n \sim \frac{5}{24}n(n+1)\binom{2n+2}{n+1}. \tag{3.12}$$

A little algebra gives

$$n(n+1)\binom{2n+2}{n+1} = 2n(n+1)(2n+1)C_n. \tag{3.13}$$

From (3.12) and (3.13) we obtain

$$E_n^{231}\mathcal{I}_n^2 = g_n \sim \frac{5}{6}n^3. \tag{3.14}$$

From (3.14) and (1.2), we conclude that

$$\mathrm{Var}_{P_n^{231}}(\mathcal{I}_n) \sim \frac{5}{6}n^3 - (\frac{\sqrt{\pi}}{2}n^{\frac{3}{2}})^2 = (\frac{5}{6} - \frac{\pi}{4})n^3,$$

which completes the proof of the theorem. □

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