THE DISTRIBUTION ON PERMUTATIONS INDUCED BY A RANDOM PARKING FUNCTION

ROSS G. PINSKY

ABSTRACT. A parking function on $[n]$ creates a permutation in S_n via the order in which the n cars appear in the n parking spaces. Placing the uniform probability measure on the set of parking functions on $[n]$ induces a probability measure on S_n. We initiate a study of some properties of this distribution. Let P_n^{park} denote this distribution on S_n and let P_n denote the uniform distribution on S_n. In particular, we obtain an explicit formula for $P_n^{\text{park}}(\sigma)$ for all $\sigma \in S_n$. Then we show that for all but an asymptotically P_n-negligible set of permutations, one has $P_n^{\text{park}}(\sigma) \in \left(\frac{(2-\epsilon)^n}{(n+1)^{n-1}}, \frac{(2+\epsilon)^n}{(n+1)^{n-1}}\right)$. However, this accounts for only an exponentially small part of the P_n^{park}-probability. We also obtain an explicit formula for $P_n^{\text{park}}(\sigma^{-1}_{n-j+1} = i_1, \sigma^{-1}_{n-j+2} = i_2, \cdots, \sigma^{-1}_n = i_j)$, the probability that the last j cars park in positions $i_1, \cdots, i_j$ respectively, and show that the j-dimensional random vector $(n+1-\sigma^{-1}_{n-j+l}, n+1-\sigma^{-1}_{n-j+2}, \cdots, n+1-\sigma^{-1}_n)$ under P_n^{park} converges in distribution to a random vector $(\sum_{r=1}^j X_r, \sum_{r=2}^j X_r, \cdots, X_{j-1}+X_j, X_j)$, where $\{X_r\}_{r=1}^j$ are IID with the Borel distribution. We then show that in fact for $j_n = o(n^{\frac{1}{6}})$, the final j_n cars will park in increasing order with probability approaching 1 as $n \to \infty$. We also obtain an explicit formula for the expected value of the left-to-right maximum statistic $X_n^{\text{LR-max}}$, which counts the total number of left-to-right maxima in a permutation, and show that $E_n^{\text{park}} X_n^{\text{LR-max}}$ grows approximately on the order $n^{\frac{1}{2}}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider a row of n parking spaces on a one-way street. A line of n cars, numbered from 1 to n, attempt to park, one at a time. The ith car's preferred space is spot number $\pi_i \in [n]$. If this space is already taken, then car i proceeds forward and parks in the first available space, if one exists.

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If the car is unable to park, it exits the street. A sequence $\pi = \{\pi_i\}_{i=1}^n$ is called a parking function on $[n]$ if all n cars are able to park. It is easy to see that π is a parking function if and only if $|\{i : \pi_i \le j\}| \ge j$, for all $j \in [n]$. Let $\mathcal{P}_n$ denote the set of parking functions. It is well-known that $|\mathcal{P}_n| = (n+1)^{n-1}$. There are a number of proofs of this result; a particularly elegant one due to Pollack can be found in [4]. There is a large literature on parking functions and their generalizations; see, for example, the survey [7].

We can consider a random parking function by placing the uniform probability measure on $\mathcal{P}_n$. Denote this probability measure by $P^{\mathcal{P}_n}$. A study of random parking functions was initiated by Diaconis and Hicks in [2]. Since each parking function yields a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, where σ_j is the number of the car that parked in space j, it follows that a random parking function induces a distribution on the set S_n of permutations of $[n]$. In this paper we initiate a study of this distribution.

We will use the notation P_n and E_n to denote the uniform probability measure and the corresponding expectation on S_n. We will denote by P_n^{park} the probability measure on S_n induced by a random parking function in $\mathcal{P}_n$. The corresponding expectation will be denoted by E_n^{park}. To be more precise concerning the definition of the induced probability measure, define $T_n : \mathcal{P}_n \to S_n$ by $T_n(\pi) = \sigma$, if when using the parking function π, σ_j is the number of the car that parked in space j, for $j \in [n]$. For example, if $n = 4$ and $\pi = 2213 \in \mathcal{P}_4$, then we have $T_4(2213) = 3124 \in S_4$. We define

$$P_n^{\text{park}}(\sigma) = P^{\mathcal{P}_n}\left(T_n^{-1}(\{\sigma\})\right). \tag{1.1}$$

For $1 \le i \le n < \infty$ and $\sigma \in S_n$, define

$$l_{n,i}(\sigma) = \max\{l \in [i] : \sigma_i = \max(\sigma_i, \sigma_{i-1}, \cdots, \sigma_{i-l+1}\}. \tag{1.2}$$

For $n \in \mathbb{N}$, define

$$L_n(\sigma) = \prod_{i=1}^n l_{n,i}(\sigma),\ \sigma \in S_n.$$

For example, if $\sigma = 379218645 \in S_9$, then $l_{n,i}(\sigma) = 1$, for $i \in \{1,4,5,7,8\}$, $l_{n,i}(\sigma) = 2$, for $i \in \{2,9\}$ and $l_{n,i}(\sigma) = 3$, for $i \in \{3,6\}$. Thus, $L_9(\sigma) = 1^5 2^2 3^2 = 36$. At certain points in the paper, it will be convenient to use the following variant of $l_{n,i}$. Define

$$\tilde{l}_{n,i}(\sigma) = l_{n,\sigma_i^{-1}}(\sigma). \tag{1.3}$$

For example, if $\sigma = 379218645$, then $\tilde{l}_{n,i}(\sigma) = 1$, for $i \in \{1,2,3,4,6\}$, $\tilde{l}_{n,i} = 2$, for $i \in \{5,7\}$ and $\tilde{l}_{n,i} = 3$, for $i \in \{8,9\}$. Of course,

$$L_n(\sigma) = \prod_{i=1}^{n} l_{n,i}(\sigma) = \prod_{i=1}^{n} \tilde{l}_{n,i}(\sigma),\ \sigma \in S_n. \tag{1.4}$$

We have the following theorem.

Theorem 1.

$$P_n^{\text{park}}(\sigma) = \frac{L_n(\sigma)}{(n+1)^{n-1}},\ \sigma \in S_n. \tag{1.5}$$

The following corollary of Theorem 1 is immediate, where the asymptotic behavior follows from Stirling's formula, $n! \sim n^n e^{-n}\sqrt{2\pi n}$.

Corollary 1. *The expected value of the random variable $L_n = L_n(\sigma)$ on (S_n, P_n) satisfies*

$$E_n L_n = \frac{1}{n!}\sum_{\sigma\in S_n} L_n(\sigma) = \frac{(n+1)^{n-1}}{n!} \sim \frac{e^{n+1}}{\sqrt{2\pi}\, n^{\frac{3}{2}}}. \tag{1.6}$$

Although $E_n L_n$ is around e^n, with high P_n-probability $L_n(\sigma)$ is near 2^n. We will prove the following weak convergence result for L_n.

Theorem 2. *For any $\epsilon \in (0,2]$, the random variable $L_n = L_n(\sigma)$ on (S_n, P_n) satisfies*

$$\lim_{n\to\infty} P_n((2-\epsilon)^n \le L_n \le (2+\epsilon)^n)) = 1. \tag{1.7}$$

The following corollary follows immediately from Theorems 1 and 2.

Corollary 2. *For any $\epsilon \in (0,2]$, the random variable $P_n^{\text{park}} = P_n^{\text{park}}(\sigma)$ on (S_n, P_n) satisfies*

$$\lim_{n\to\infty} P_n\left(\left\{\sigma \in S_n : P_n^{\text{park}}(\sigma) \in \left(\frac{(2-\epsilon)^n}{(n+1)^{n-1}}, \frac{(2+\epsilon)^n}{(n+1)^{n-1}}\right)\right\}\right) = 1. \tag{1.8}$$

And the following corollary follows immediately from Theorem 1, Corollary 1 and Stirling's formula.

Corollary 3. *The expectation of the random variable $P_n^{\text{park}} = P_n^{\text{park}}(\sigma)$ on (S_n, P_n) is given by*

$$E_n P_n^{\text{park}} = \frac{1}{n!} \sim \frac{e^n}{\sqrt{2\pi} n^{n+\frac{1}{2}}}. \tag{1.9}$$

Comparing (1.8) and (1.9), we see that for all but an asymptotically P_n-negligible set of permutations in S_n, the P_n^{park}-probability of a permutation in S_n is approximately $\frac{2^n}{(n+1)^{n-1}}$, but the "average" P_n^{park}-probability of a permutation in S_n is exponentially larger, namely asymptotic to $\frac{e^n}{\sqrt{2\pi}n^{n+\frac{1}{2}}}$. By Stirling's formula, it also follows that there is a set $A_n \subset S_n$ with $P_n(A_n) \to 1$ for which $P_n^{\text{park}}(A_n)$ is around $(\frac{2}{e})^n$. There is an asymptotically P_n-negligible set of permutations in S_n each of whose elements has super-exponentially larger P_n^{park}-probability than the average probability, and an asymptotically P_n-negligible set of permutations in S_n each of whose elements has exponentially smaller P_n^{park}-probability than the average probability. In particular, we have the following corollary.

Corollary 4. *The maximum value of* $P_n^{\text{park}} = P_n^{\text{park}}(\sigma)$ *is equal to* $\frac{n!}{(n+1)^{n-1}} \sim \frac{\sqrt{2\pi}\, n^{\frac{3}{2}}}{e^{n+1}}$ *and is attained uniquely at* $\sigma = 1\cdots n$. *The minimum value of* P_n^{park} *is equal to* $\frac{1}{(n+1)^{n-1}}$ *and is attained uniquely at* $\sigma = n\cdots 1$.

Proof. The function $L_n = L_n(\sigma), \sigma \in S_n$, attains its maximum value $n!$ uniquely at $\sigma = 1\cdots n$ and attains its minimum value 1 uniquely at $\sigma = n\cdots 1$. . □

Let σ^{-1} denote the inverse permutation of σ. So $\sigma_k^{-1} = j$ if and only if $\sigma_j = k$. In car parking language,$\sigma_k^{-1} = j$ means that car number k parked in space number j. From the definition of a parking function, it is obvious that

$$P_n^{\text{park}}(\sigma_1^{-1} = j) = P^{\mathcal{P}_n}(\pi_1 = j),\ j \in [n].$$

In [2], the following asymptotic behavior was proven for π_1 (or any π_k by symmetry):

(1.10)

For fixed $j, P^{\mathcal{P}_n}(\pi_1 = j) \sim \dfrac{1 + P(X \ge j)}{n}$;

For fixed $j, P^{\mathcal{P}_n}(\pi_1 = n - j) \sim \dfrac{P(X \le j+1)}{n}$,

where X is a random variable satisfying $P(X = j) = e^{-j}\dfrac{j^{j-1}}{j!},\ j = 1, 2, \cdots$.

The distribution $\{e^{-j}\frac{j^{j-1}}{j!}\}_{j=1}^{\infty}$ is called the Borel distribution. It is not obvious that it is a distribution, that is, that it sums to 1. For more on this, see [2, p.135]. It follows that (1.10) also holds with $P^{\mathcal{P}_n}$ and π_1 replaced respectively by P_n^{park} and σ_1^{-1}. It would be nice to obtain asymptotic results for $P_n^{\text{park}}(\sigma_{j_n}^{-1} = k_n)$, for general j_n, k_n. We pursue this direction when j_n is near n, that is for the last cars to park.

Theorem 3.

(1.11)

$$P_n^{park}(\sigma_{n-j+1}^{-1} = i_1, \sigma_{n-j+2}^{-1} = i_2, \cdots, \sigma_n^{-1} = i_j) = \frac{(n-j)! \left(\prod_{l=1}^{j} (k_l - k_{l-1})^{k_l - k_{l-1} - 2}\right) (n - k_j + 1)^{n-k_j-1} \prod_{l=1}^{j} \tilde{l}_{n,n-j+l}(\sigma)}{(n+1)^{n-1} \left(\prod_{l=1}^{j} (k_l - k_{l-1} - 1)!\right) (n - k_j)!},$$

where $\{k_l\}_{l=1}^{j}$ *is the increasing rearrangement of* $\{i_l\}_{l=1}^{j}$, $k_0 = 0$ *and* $\tilde{l}_{n,\cdot}(\sigma)$ *is as in* (1.3).

One can use Theorem 3 to make explicit calculations. Here we treat the cases $j = 1$ and $j = 2$ in depth in Corollaries 5-8. This will lead us to a result for general j in Corollary 9 and for growing $j = j_n$ in Theorem 4.

We begin with the case $j = 1$.

Corollary 5.

$$P_n^{park}(\sigma_n^{-1} = k) = \frac{1}{n} k^k \binom{n}{k} \frac{(n-k+1)^{n-k-1}}{(n+1)^{n-1}}, \ k \in [n]. \tag{1.12}$$

Proof. Immediate from (1.11), noting that if $\sigma_n^{-1} = k$, then $\tilde{l}_{n,n}(\sigma) = k$. □

From Corollary 5 we obtain the following asymptotic formulas.

Corollary 6. *i. For fixed* $m \in \mathbb{N}$,

$$\lim_{n\to\infty} P_n^{park}(\sigma_n^{-1} = n + 1 - m) = \frac{m^{m-1} e^{-m}}{m!}. \tag{1.13}$$

Thus, the random variable $n+1-\sigma_n^{-1}$ *under* P_n^{park} *converges in distribution to a random variable* X *with the Borel distribution.*

ii. For fixed $k \in \mathbb{N}$,

$$P_n^{park}(\sigma_n^{-1} = k) \sim \frac{k^k e^{-k}}{k!} \frac{1}{n}, \ \text{as } n \to \infty. \tag{1.14}$$

iii. Let $c_n n$ *be an integer with* $\lim_{n\to\infty} c_n = c \in (0, 1)$. *Then*

$$P_n^{park}(\sigma_n^{-1} = c_n n) \sim \frac{1}{(2\pi c)^{\frac{1}{2}} (1-c)^{\frac{3}{2}}} \frac{1}{n^{\frac{3}{2}}}. \tag{1.15}$$

Proof. The proof follows from (1.12) and standard asymptotic analysis. We write out the proof of part (iii). Substituting in (1.12), we have

(1.16)
$$P_n^{\text{park}}(\sigma_n^{-1}=c_nn)=\frac{1}{n}(c_nn)^{c_nn}\frac{n!}{(c_nn)!((1-c_n)n)!}\frac{(n-c_nn+1)^{n-c_nn-1}}{(n+1)^{n-1}}.$$

Replacing the three factorials on the right hand side of (1.16) by their Stirling's formula approximations, $m!\sim m^me^{-m}\sqrt{2\pi m}$ as $m\to\infty$, and performing many cancelations, we have

$$\frac{1}{n}(c_nn)^{c_nn}\frac{n!}{(c_nn)!((1-c_n)n)!}\frac{(n-c_nn+1)^{n-c_nn-1}}{(n+1)^{n-1}}\sim$$
$$\frac{n^{n-1}\left(1+\frac{1}{n-c_nn}\right)^{n-c_nn-1}}{n\sqrt{c_n}(1-c_n)\sqrt{2\pi(1-c_n)n}\,(n+1)^{n-1}}.$$

Since $\lim_{n\to\infty}\frac{n^{n-1}\left(1+\frac{1}{n-c_nn}\right)^{n-c_nn-1}}{(n+1)^{n-1}}=1$, we conclude that the right hand side of (1.16) is asymptotic to $\frac{1}{(2\pi c)^{\frac12}(1-c)^{\frac32}}\frac{1}{n^{\frac32}}$. □

Remark. One can check that $\frac{k^ke^{-k}}{k!}$ is decreasing in k. Thus, from part (ii), for fixed k, $P_n^{\text{park}}(\sigma_n^{-1}=k)$ is on the order $\frac{1}{n}$ and decreasing in k. From part (iii), the probability of σ_n^{-1} taking on any particular value in the bulk is even smaller, namely on the order $n^{-\frac32}$. And from part (i), the distance of σ_n^{-1} from n converges in distribution as $n\to\infty$.

We now turn to the case $j=2$.

Corollary 7. *Let*

(1.17)
$$A_n(a,b)=\frac{(n-2)!a^{a-2}(b-a)^{b-a-2}(n-b+1)^{n-b-1}}{(a-1)!(b-a-1)!(n-b)!(n+1)^{n-1}},\ \textit{for } 1\le a<b\le n.$$

Then

(1.18)
$$P_n^{park}(\sigma_{n-1}^{-1}=l,\sigma_n^{-1}=m)=\begin{cases}lmA_n(l,m), & \textit{if } 1\le l<m\le n;\\ (l-m)mA_n(m,l), & \textit{if } 1\le m<l\le n.\end{cases}.$$

Proof. Immediate from (1.11), noting that if $\sigma_{n-1}^{-1}=l$ and $\sigma_n^{-1}=m$, then

$$\begin{cases}\tilde l_{n,n-1}(\sigma)=l,\ \tilde l_{n,n}(\sigma)=m, & \text{if } l<m;\\ \tilde l_{n,n-1}(\sigma)=l-m,\ \tilde l_{n,n}(\sigma)=m, & \text{if } l>m.\end{cases}$$

□

From Corollary 7 we obtain the following asymptotic formulas.

Corollary 8. *i. For fixed* l, m,

(1.19)
$$\lim_{n\to\infty} P_n^{park}(\sigma_{n-1}^{-1} = n+1-l, \sigma_n^{-1} = n+1-m) = \frac{(l-m)^{l-m-1}m^{m-1}}{(l-m)!m!}e^{-l},\ 1 \le m < l;$$

(1.20)
$$P_n^{park}(\sigma_{n-1}^{-1} = n+1-l, \sigma_n^{-1} = n+1-m) \sim \frac{(m-l)^{m-l}l^{l-1}}{l!(m-l)!}\frac{1}{n},\ 1 \le l < m.$$

Thus, the random vector $(n+1-\sigma_{n-1}^{-1}, n+1-\sigma_n^{-1})$ *under* P_n^{park} *converges in distribution to a random vector* $(X_1 + X_2, X_2)$ *where* X_1 *and* X_2 *are IID with the Borel distribution.*
ii. For fixed, l, m,

$$(1.21)\quad P_n^{park}(\sigma_{n-1}^{-1} = l, \sigma_n^{-1} = m) \sim \begin{cases} \frac{l^l}{l!}\frac{(m-l)^{m-l-1}m}{(m-l)!}e^{-m}\frac{1}{n^2},\ 1 \le l < m; \\ \frac{m^m}{m!}\frac{(l-m)^{l-m}}{(l-m)!}e^{-l}\frac{1}{n^2},\ 1 \le m < l. \end{cases}$$

Proof. Standard asymptotic analysis. □

Part (i) of Corollaries 6 and 8 lead us to the following result for all j, which we will prove.

Corollary 9. *Let* $j \in \mathbb{N}$. *The* j*-dimensional random vector* $(n+1-\sigma_{n-j+l}^{-1}, n+1-\sigma_{n-j+2}^{-1}, \cdots, n+1-\sigma_n^{-1})$ *under* P_n^{park} *converges in distribution to a random vector* $(\sum_{r=1}^j X_r, \sum_{r=2}^j X_r, \cdots, X_{j-1}+X_j, X_j)$, *where* $\{X_r\}_{r=1}^j$ *are IID with the Borel distribution.*
In particular then, for any $j \in \mathbb{N}$, *the last* j *cars from among the* n *cars will park in increasing order with probability approaching 1 as* $n \to \infty$:

$$(1.22)\qquad \lim_{n\to\infty} P_n^{park}(\sigma_{n+1-j}^{-1} < \sigma_{n+1-j+2}^{-1} < \cdots < \sigma_n^{-1}) = 1.$$

In fact, we can extend (1.22) to the last j_n cars, where $j_n = o(n^{\frac{1}{6}})$. We will prove the following result.

Theorem 4. *For* $j_n = o(n^{\frac{1}{6}})$, *the last* j_n *cars from among the* n *cars will park in increasing order with probability approaching 1 as* $n \to \infty$:

$$(1.23)\qquad \lim_{n\to\infty} P_n^{park}(\sigma_{n+1-j_n}^{-1} < \sigma_{n+1-j_n+1}^{-1} < \cdots < \sigma_n^{-1}) = 1.$$

We now consider the left-to-right maximum statistic. Recall that a position $i \in [n]$ is called a *left-to-right-maximum* for the permutation $\sigma \in S_n$ if $\sigma_i > \sigma_k$, for all $k \in [i-1]$. Let $X_n^{\text{LR-max}} = X_n^{\text{LR-max}}(\sigma)$ denote the left-to-right maximum statistic, that is, the total number of left-to-right maxima in $\sigma \in S_n$. Under the uniform probability measure P_n on S_n, it follows from symmetry that the probability that i is a left-to-right maximum is $\frac{1}{i}$; thus $E_n X_n^{\text{LR-max}} = \sum_{j=1}^n \frac{1}{j} \sim \log n$. It is well-known that under the uniform distribution, the left-to-right maximum statistic has the same distribution as the cycle statistic that counts the total number of cycles in a permutation [1]. The well-known law of large numbers and central limit theorem for the cycle statistic under the uniform distribution thus also holds for the left-to-right maximum statistic.

The left-to-right maximum statistic behaves very differently under P_n^{park}. Note that it is immediate from Theorem 4 that $E_n^{\text{park}} X_n^{\text{LR-max}} \ge \omega_n$, for large n, if $\omega_n = o(n^{\frac{1}{6}})$. In fact, we shall see that $E_n^{\text{park}} X_n^{\text{LR-max}}$ grows on an order at least $n^{\frac{1}{2}}$ and no more than $n^{\frac{1}{2}+\epsilon}$, for any $\epsilon > 0$. The theorem below gives an exact formula for the probability that i is a left-to-right maximum and that also $\sigma_i = j$, from which an exact formula for $E_n^{\text{park}} X_n^{\text{LR-max}}$ follows. (Of course, the probability that i is a left-to-right maximum and that $\sigma_i = j$ is equal to zero if $i > j$.)

Theorem 5. *i.*

(1.24)
$$P_n^{park}(i \text{ is a left-to-right maximum and } \sigma_i = j) =$$
$$\binom{j-1}{i-1} i^{i-1}(n-i+1)^{j-i-1}\left(i(n-j)(n+1)^{-j} + (n+1)^{1-j}\right), \ 1 \le i \le j \le n.$$

ii.

(1.25)
$$E_n^{park} X_n^{LR\text{-}max} =$$
$$\sum_{j=1}^n \sum_{i=1}^j \binom{j-1}{i-1} i^{i-1}(n-i+1)^{j-i-1}\left(i(n-j)(n+1)^{-j} + (n+1)^{1-j}\right).$$

We didn't find the right hand side of (1.25) very amenable to direct asymptotic analysis. However, we were able to express (1.25) in a different form that is more tractable for such analysis. We have the following theorem.

Theorem 6. *i.*

$$E_n^{park} X_n^{LR\text{-}max} = n - \sum_{l=1}^{n} \frac{n-l}{l(l+1)} \frac{\prod_{j=0}^{l-1}(n-j)}{(n+1)^l}. \tag{1.26}$$

ii.

$$\begin{aligned} &\limsup_{n\to\infty} \frac{E_n^{park} X_n^{LR\text{-}max}}{n^{\frac{1}{2}+\epsilon}} = 0, \ \textit{for any } \epsilon > 0; \\ &\liminf_{n\to\infty} \frac{E_n^{park} X_n^{LR\text{-}max}}{n^{\frac{1}{2}}} > 0. \end{aligned} \tag{1.27}$$

The proof of Theorem 1 is given in section 2 and the proof of Theorem 2 is given in section 3. The proof of Theorem 3 is given in section 4, the proof of Corollary 9 is given in section 5 and the proof of Theorem 4 is given in section 6. The proof of Theorem 5 is given in section 7 and the proof of Theorem 6 is given in sections 8 and 9.

2. Proof of Theorem 1

Recall the definitions of $l_{n,i}$ and $\tilde{l}_{n,i}$ in (1.2) and (1.3). For the proof of the theorem, it will be convenient to work with $\tilde{l}_{n,i}$. Recall from (1.4) that

$$L_n(\sigma) = \prod_{i=1}^{n} \tilde{l}_{n,i}(\sigma).$$

The theorem will follow if we show that for each $\sigma \in S_n$, there are $L_n(\sigma)$ different parking functions $\pi \in \mathcal{P}_n$ such that $T_n(\pi) = \sigma$, where T_n is as in the paragraph containing equation (1.1). Before giving a formal proof of the theorem, we illustrate the proof with a concrete example, from which the general result should be clear. Consider the permutation $\sigma = 379218645 \in S_9$. We look for those $\pi \in \mathcal{P}_9$ that satisfy $T_9(\pi) = \sigma$. From the definition of the parking process and from the definition of T_n, we need $\pi_1 = 5$ in order to have $\sigma_5 = 1$, $\pi_2 = 4$ in order to have $\sigma_4 = 2$, $\pi_3 = 1$ in order to have $\sigma_1 = 3$ and $\pi_4 = 8$ in order to have $\sigma_8 = 4$. In order to have $\sigma_9 = 5$, we can either have $\pi_5 = 9$, in which case car number 5 parks in its preferred space 9, or alternatively, $\pi_5 = 8$, in which case car number 5 attempts to park in its preferred space 8 but fails, and then moves on to space 9 and parks. Then we need $\pi_6 = 7$ in order to have $\sigma_7 = 6$. Then similar to the explanation regarding π_5, we need π_7 to be either 1 or 2 in order to have $\sigma_2 = 7$. In order to have $\sigma_6 = 8$, we can have either $\pi_8 = 6$, in which car number 8 parks directly in its preferred space 6, or alternatively $\pi_8 = 5$, in which case car number 8 tries and fails to park in space number 5 and then parks in space number 6, or alternatively, $\pi_8 = 4$, it which case car number 8 tries and fails to park in space number 4 and then also in space number 5, before finally parking in space number 6. Similarly, we need π_9 to be equal to 1,2 or 3 in order to have $\sigma_9 = 3$. Thus, there are $1 \times 1 \times 1 \times 1 \times 2 \times 1 \times 2 \times 3 \times 3 = \prod_{i=1}^{9} \tilde{l}_{9,i}(\sigma) = L_9(\sigma)$ different parking functions $\pi \in \mathcal{P}_9$ that yield $T_9(\pi) = \sigma$.

To give a formal proof for the general case, fix $\sigma \in S_n$. In order to have $T_n(\pi) = \sigma$, first we need $\pi_1 = \sigma_1^{-1}$. Thus there is just one choice for π_1, and note that $\tilde{l}_{n,1}(\sigma) = 1$. Now let $k \in [n-1]$ and assume that we have chosen $\pi_1, \cdots \pi_k$ in such a way that car number i has parked in space σ_i^{-1}, for $i \in [k]$. We now want car number $k+1$ to park in space σ_{k+1}^{-1}. By construction, this space is vacant at this point, and so are the $\tilde{l}_{n,k+1}(\sigma) - 1$ spaces immediately to the left of this space. However the space $\tilde{l}_{n,k+1}$ spaces to the left of this space is not vacant (or possibly this space doesn't exist–it would be the zeroth space). Thus, by the parking process, car number $k+1$ will park in space σ_{k+1}^{-1} if and only if π_{k+1} is equal to one of the $\tilde{l}_{n,k+1}(\sigma)$ numbers $\sigma_{k+1}^{-1}, \sigma_{k+1}^{-1} - 1, \cdots, \sigma_{k+1}^{-1} - \tilde{l}_{n,k+1}(\sigma) + 1$. This shows that there are $L_n(\sigma) = \prod_{i=1}^{n} \tilde{l}_{n,i}(\sigma)$ different parking functions π satisfying $T_n(\pi) = \sigma$. □

3. Proof of Theorem 2

We begin with several preliminary results. Recall that P_n is the uniform probability measure on S_n.

Lemma 1.

$$P_n(l_{n,i} = j) = \begin{cases} \frac{1}{j} - \frac{1}{j+1} = \frac{1}{j(j+1)}, \ j = 1, \cdots, i-1; \\ \frac{1}{i}, \ j = i. \end{cases} \tag{3.1}$$

Proof. Fix i and let $j \in [i]$. The event $\{l_{n,i}(\sigma) \geq j\}$ is the event $\big\{\sigma_i = \max\{\sigma_i, \sigma_{i-1}, \cdots, \sigma_{i-j+1}\}\big\}$. Since P_n is the uniform distribution on S_n, we have

$$P_n(l_{n,i} \geq j) = \frac{1}{j}, \ i \in [n], \ 1 \leq j \leq i. \tag{3.2}$$

The lemma now follows. □

We now write

$$\mathcal{S}_n := \log L_n = \sum_{i=1}^{n} \log l_{n,i}. \tag{3.3}$$

From Lemma 1, we have

$$E_n \log l_{n,i} = \sum_{j=1}^{i-1} \frac{\log j}{j(j+1)} + \frac{\log i}{i}. \tag{3.4}$$

Note that $E_n \log l_{n,i}$ does not depend on n, but of course it is only defined for $1 \le i \le n$.

Lemma 2.

$$\lim_{n,i\to\infty} E_n \log l_{n,i} = \log 2. \tag{3.5}$$

Proof. Recall the Abel-type summation formula [5]:

$$\sum_{1<r\le x} a(r)f(r) = A(x)f(x) - A(1)f(1) - \int_1^x A(t)f'(t)dt, \text{ where } A(r) = \sum_{i=1}^{r} a_i.$$

We apply this formula with $a(r) = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$ and $f(r) = \log r$. We have $A(r) = 1 - \frac{1}{r+1} = \frac{r}{r+1}$. Recalling (3.4), we obtain

$$\lim_{n,i\to\infty} E_n \log l_{n,i} = \lim_{i\to\infty} \sum_{j=1}^{i-1} \frac{\log j}{j(j+1)} = \lim_{i\to\infty} \left(\frac{i}{i+1} \log i - \int_1^i \frac{t}{t+1}\frac{1}{t} dt \right) =$$

$$\lim_{i\to\infty} \left(\frac{i}{i+1} \log i - \log(i+1) + \log 2 \right) = \lim_{i\to\infty} \left(\log \frac{i}{i+1} - \frac{\log i}{i+1} + \log 2 \right) = \log 2.$$

□

From (3.3) and (3.5), we conclude that

$$\lim_{n\to\infty} \frac{E_n \mathcal{S}_n}{n} = \lim_{n\to\infty} \frac{1}{n} E_n \log L_n = \log 2. \tag{3.6}$$

We now consider $E_n \mathcal{S}_n^2$. We have

(3.7)

$$E_n \mathcal{S}_n^2 = E_n \left(\sum_{i=1}^{n} \log l_{n,i} \right)^2 = \sum_{i=1}^{n} E_n \log l_{n,i}^2 + 2 \sum_{1\le i<j\le n} E_n \log l_{n,i} \log l_{n,j}.$$

We have the following proposition.

Proposition 1. *For $1 \le i < j \le n$, the random variables $l_{n,i}$ and $l_{n,j}$ on (S_n, P_n) are negatively correlated; that is,*

$$P_n(l_{n,i} \ge k, l_{n,j} \ge k) \le P_n(l_{n,i} \ge k)P_n(l_{n,j} \ge l), \text{ for } k,l \ge 1. \tag{3.8}$$

Proof. Since P_n is the uniform probability measure on S_n, for any $k \le i$, the events $\{l_{n,i} \ge k\} = \big\{\sigma_i = \max(\sigma_i, \cdots, \sigma_{i-k+1})\big\}$ and $\{l_{n,j} \ge l\} = \big\{\sigma_j = \max(\sigma_j, \cdots, \sigma_{j-l+1})\big\}$ are independent if $l \le j - i$. Thus, (3.8) holds with equality in these cases.

Consider now the case $k \le i$ and $j - i + 1 \le l \le j$. In this case

$$\{l_{n,i} \ge k, l_{n,j} \ge l\} = \big\{\sigma_j = \max(\sigma_j, \sigma_{j-1}, \cdots, \sigma_r)\big\} \cap \big\{\sigma_i = \max(\sigma_i, \sigma_{i-1}, \cdots, \sigma_{i-k+1})\big\}, \tag{3.9}$$

where $r = \min(i - k + 1, j - l + 1)$.

We have

$$P_n\left(\sigma_j = \max(\sigma_j, \sigma_{j-1}, \cdots, \sigma_r)\right) = \frac{1}{\max(l, j - i + k)} \le \frac{1}{l}. \tag{3.10}$$

Also,

$$P_n\left(\sigma_i = \max(\sigma_i, \sigma_{i-1}, \cdots, \sigma_{i-k+1} | \sigma_j = \max(\sigma_j, \sigma_{j-1}, \cdots, \sigma_r)\right) = \frac{1}{k}, \tag{3.11}$$

because $\{\sigma_i, \cdots, \sigma_{i-k+1}\} \subset \{\sigma_j, \sigma_{j-1}, \cdots, \sigma_r\}$. The proposition follows from (3.9)–(3.11) and (3.2). □

We can now prove the theorem.

Proof of Theorem 2. Since $l_{n,i}$ and $l_{n,j}$ are negatively correlated, one has $E_n f(l_{n,i})g(l_{n,j}) \le E_n f(l_{n,i})E_n g(l_{n,j})$, if f and g are increasing functions on $[n]$. In particular then,

$$E_n \log l_{n,i} \log l_{n,j} \le E_n \log l_{n,i} E_n \log l_{n,j}. \tag{3.12}$$

From (3.3) and (3.12), a standard straightforward calculation gives

$$\text{Var}(\mathcal{S}_n) \le \sum_{i=1}^{n} \text{Var}(\log l_{n,i}). \tag{3.13}$$

From (3.1), we have

$$E_n(\log l_{n,i})^2 = \sum_{j=1}^{i-1} \frac{(\log j)^2}{j(j+1)} + \frac{(\log i)^2}{i(i+1)}.$$

Using this with (3.5) and (3.13), we conclude that there exists a $C > 0$ such that

$$\text{Var}(\mathcal{S}_n) \le Cn, \ n \in \mathbb{N}. \tag{3.14}$$

From (3.6) and(3.14), it follows from the second moment method (Chebyshev's inequality) that

$$\lim_{n\to\infty} P_n(\log 2 - \epsilon \le \frac{\mathcal{S}_n}{n} \le \log 2 + \epsilon) = 1, \text{ for all } \epsilon > 0. \tag{3.15}$$

Now (1.7) follows from (3.15) and (3.3). □

4. Proof of Theorem 3

We count how many parking functions $\{\pi_l\}_{l=1}^n$ yield the event $\{\sigma^{-1}_{n-j+1} = i_1, \sigma^{-1}_{n-j+2} = i_2, \cdots, \sigma^{-1}_n = i_j\}$. In order for this event to occur, the first $n-j$ cars have to park arbitrarily in the spots $[n] - \{k_l\}_{l=1}^j$. Then the last j cars have to park, with each one in its appropriate space.

We now analyze the parking of the first $n-j$ cars. In order for $k_1 - 1$ cars from among the first $n-j$ cars to park in the first $k_1 - 1$ spaces, but for no one of the first $n-j$ cars to park in the k_1th space, it follows from the definition of the parking process that there must be a collection of $k_1 - 1$ of the $\{\pi_i\}_{l=1}^{n-j}$ which constitute a parking function of length $k_1 - 1$. We can freely choose which of the first $n-j$ cars to use for these spaces. Since there are $k_1^{k_1-2}$ parking functions of length $k_1 - 1$, this gives $\binom{n-j}{k_1-1} k_1^{k_1-2}$ different choices.

In order for $k_2 - k_1 - 1$ cars, from among the $n-j-(k_1-1)$ cars that still remain from among the first $n-j$ cars, to park in the spaces $k_1+1, \cdots, k_2-1$, but for no one of the first $n-j$ cars to park in the k_1th space or the k_2th space, there must be a collection of k_2-k_1-1 of the remaining $n-j-(k_1-1)$ members of $\{\pi_l\}_{l=1}^{n-j}$ that constitute a parking function of length k_2-k_1-1, but shifted forward by k_1 spaces. (If $\{\pi_l\}_{l=1}^m$ is a parking function of length m, then we call the collection $\{a + \pi_l\}_{l=1}^m$, where $a \in \mathbb{N}$, a parking function of length m shifted forward by a spaces. Obviously, the number of such parking functions coincides with the number of parking functions of length m.) We can freely choose which of the remaining $n-j-(k_1-1)$ cars from among the first $n-j$ cars to use for these spaces. Thus, there are $\binom{n-j-(k_1-1)}{k_2-k_1-1}(k_2-k_1)^{k_2-k_1-2}$ different choices.

We continue counting in this fashion until we have chosen cars for the $k_j - k_{j-1} - 1$ spaces between spaces k_{j-1} and k_j. Recall that k_0 has been defined to be 0. Now we need for all of the $n-j-\sum_{i=1}^{j}(k_i-k_{i-1}-1) = n-k_j$ cars that remain from among the first $n-j$ cars to park in the spaces $k_j+1, \cdots, n$. In order for this to occur, but for no one of the first $n-j$ cars to park in the k_jth space, these remaining members of $\{\pi_i\}_{l=1}^{n-j}$ must constitute a parking function of length $n-k_j$, but shifted forward by k_j spaces. This gives $(n-k_j+1)^{n-k_j-1}$ different choices.

Thus, we have shown that the number of ways to choice $\{\pi_l\}_{l=1}^{n-j}$ is

$$\binom{n-j}{k_1-1} k_1^{k_1-2} \binom{n-j-k_1+1}{k_2-k_1-1} (k_2-k_1)^{k_2-k_1-2} \cdots \times \binom{n-j-\sum_{l=1}^{j-1}(k_l-k_{l-1})}{k_j-k_{j-1}-1} (k_j-k_{j-1})^{k_j-k_{j-1}-2}(n-k_j+1)^{n-k_j-1} = \frac{(n-j)!}{(n-k_j)!\prod_{l=1}^{j}(k_l-k_{l-1}-1)!}(n-k_j+1)^{n-k_j-1}\prod_{l=1}^{j}(k_l-k_{l-1})^{k_l-k_{l-1}-2}. \tag{4.1}$$

Now we count how many ways we can choice the $\{\pi_l\}_{l=n-j+1}^{n}$ in order that $\sigma_{n-j+l}^{-1} = i_l,\ l = 1, \cdots, j$, or equivalently, in order that car $n-j+l$ park in space $i_l,\ l = 1, \cdots, j$. This calculation uses the same reasoning that was used in the proof of Theorem 1. After the first $n-j$ cars have parked appropriately as above, the spaces $\{k_l\}_{l=1}^{j} = \{i_l\}_{l=1}^{j}$ are still vacant. From the definition of the parking process and from the definition of $\tilde{l}_{n,n-j+1}(\sigma)$, in order that car $n-j+1$ park in space i_1, π_{n-j+1} must take on one of the values $i_1.i_1-1, \cdots i_1-\tilde{l}_{n,n-j+1}(\sigma)+1$. Then in order that car $n-j+2$ park in space i_2, π_{n-j+2} must take on one of the values $i_2, i_2-1, \cdots i_2-\tilde{l}_{n,n-j+2}(\sigma)+1$. Continuing like this, we conclude that there are $\prod_{l=1}^{j}\tilde{l}_{n,n-j+l}(\sigma)$ choices. Using this with (4.1), and noting that there are $(n+1)^{n-1}$ different parking functions of length n, we obtain (1.11). □

5. Proof of Corollary 9

Let $\{m_r\}_{r=1}^{j} \subset \mathbb{N}$. By Theorem 3, for $n \ge \sum_{r=1}^{j} m_r$, the probability $P_n^{\text{park}}(n+1-\sigma_{n-j+l}^{-1} = \sum_{r=l}^{j} m_r;\ l = 1, \cdots, j)$ is given by the formula on the right hand side of (1.11) with with $i_l = n+1-\sum_{r=l}^{j} m_r,\ l \in [j]$. Since these $\{i_l\}_{l=1}^j$ are increasing, we have $k_l = i_l,\ l \in [j]$. So we have

$$(5.1)\qquad \begin{aligned} &k_l - k_{l-1} = m_{l-1},\ l = 2, \cdots j;\\ &k_1 - k_0 = n + 1 - \sum_{r=1}^{j} m_r. \end{aligned}$$

Since the $\{i_l\}_{l=1}^j$ are increasing and represent the positions respectively of the numbers $\{n-j+l\}_{l=1}^j$ in the permutation σ, it follows from the definition of $\tilde{l}_{n,\cdot}(\sigma)$ in (1.3) that

$$(5.2)\qquad \tilde{l}_{n,n-j+l}(\sigma) = i_l = n + 1 - \sum_{r=l}^{j} m_r.$$

Substituting from (5.1) and (5.2) into the right hand side of (1.11), the expression there becomes

(5.3)

$$\frac{(n-j)!\left(n+1-\sum_{r=1}^j m_r\right)^{n-1-\sum_{r=1}^j m_r}\left(\prod_{l=2}^j m_{l-1}^{m_{l-1}-2}\right) m_j^{m_j-2}\prod_{l=1}^j\left(n+1-\sum_{r=l}^j m_r\right)}{(n+1)^{n-1}(n-\sum_{r=1}^j m_r)!\left(\prod_{l=2}^j (m_{l-1}-1)!\right)(m_j-1)!}.$$

We rewrite the terms in (5.3) that do not involve n as

$$(5.4)\qquad \frac{\left(\prod_{l=2}^j m_{l-1}^{m_{l-1}-2}\right) m_j^{m_j-2}}{\left(\prod_{l=2}^j (m_{l-1}-1)!\right)(m_j-1)!} = \prod_{l=1}^{j} \frac{m_l^{m_l-1}}{m_l!}.$$

The terms in (5.3) that involve n satisfy

(5.5)

$$\lim_{n\to\infty} \frac{(n-j)!\left(n+1-\sum_{r=1}^j m_r\right)^{n-1-\sum_{r=1}^j m_r}\prod_{l=1}^j\left(n+1-\sum_{r=l}^j m_r\right)}{(n+1)^{n-1}(n-\sum_{r=1}^j m_r)!} = e^{-\sum_{r=1}^j m_r}.$$

From (5.3)-(5.5), it follows that the right hand side of (1.11) with $i_l = n+1-\sum_{r=l}^j m_r,\ l \in [j]$, converges as $n \to \infty$ to $\prod_{l=1}^j \frac{m_l^{m_l-1}e^{-m_l}}{m_l!}$. Thus, we have proven that

$$(5.6)\quad \lim_{n\to\infty} P_n^{\text{park}}(n+1-\sigma^{-1}_{n-j+l} = \sum_{r=l}^{j} m_r;\ l = 1, \cdots, j) = \prod_{l=1}^{j} \frac{m_l^{m_l-1}e^{-m_l}}{m_l!}.$$

The corollary follows from this. □

6. Proof of Theorem 4

From the proof of Corollary 9 up to (5.4), in particular, from (5.3) and (5.4), we have for $j_n \le n$,

(6.1)

$$P_n^{\text{park}}\left(n+1-\sigma^{-1}_{n-j_n+1}=\sum_{r=1}^{j_n} m_r, n+1-\sigma^{-1}_{n-j_n+2}=\sum_{r=2}^{j_n} m_r,\cdots,n+1-\sigma^{-1}_n=m_{j_n}\right)=$$
$$\left(\prod_{r=1}^{j_n}\frac{m_r^{m_r-1}e^{-m_r}}{m_r!}\right)A(n,j_n;m_1\cdots,m_{j_n})B(n,j_n;m_1,\cdots,m_{j_n}),\ \text{for}\ \sum_{r=1}^{j_n} m_r\le n,\ m_r\ge 1,$$

where

(6.2)

$$A(n,j;m_1\cdots,m_j)=e^{\sum_{r=1}^j m_r}\left(\frac{n+1-\sum_{r=1}^j m_r}{n+1}\right)^{n-1}=e^{\sum_{r=1}^j m_r}\left(1-\frac{\sum_{r=1}^j m_r}{n+1}\right)^{n-1};$$
$$B(n,j;m_1,\cdots,m_j)=\frac{(n-j)!}{(n-\sum_{r=1}^j m_r)!}\frac{\prod_{r=1}^j(n+1-\sum_{r=l}^j m_r)}{(n+1-\sum_{r=1}^j m_r)^{\sum_{r=1}^j m_r}}.$$

By Stirling's formula, $\frac{m^{m-1}e^{-m}}{m!}\sim\frac{1}{\sqrt{2\pi}}m^{-\frac{3}{2}}$. Thus, there exists a $C>0$ such that

$$\sum_{m=N+1}^{\infty}\frac{m^{m-1}e^{-m}}{m!}<\frac{C}{N^{\frac12}},\ N\in\mathbb{N}.$$

Since $\{\frac{m^{m-1}e^{-m}}{m!}\}_{m=1}^\infty$ is a probability distribution, it then follows that

$$\sum_{m=1}^{N}\frac{m^{m-1}e^{-m}}{m!}\ge 1-\frac{C}{N^{\frac12}},\ N\in\mathbb{N}. \tag{6.3}$$

Using (6.3) for the inequality below, we have

$$\sum_{\substack{1\le m_r\le N\\ r=1,\cdots,j_n}}\prod_{r=1}^{N}\frac{m_r^{m_r-1}e^{-m_r}}{m_r!}=\left(\sum_{m=1}^{N}\frac{m^{m-1}e^{-m}}{m!}\right)^{j_n}\ge\left(1-\frac{C}{N^{\frac12}}\right)^{j_n}. \tag{6.4}$$

Substituting N_n for N in (6.4), a standard estimate gives

$$\lim_{n\to\infty}\sum_{\substack{1\le m_r\le N_n\\ r=1,\cdots,j_n}}\prod_{r=1}^{N_n}\frac{m_r^{m_r-1}e^{-m_r}}{m_r!}=1,\ \text{if}\ j_n=o(N_n^{\frac12}). \tag{6.5}$$

We can express $P_n^{\text{park}}(\sigma^{-1}_{n+1-j_n} < \sigma^{-1}_{n+1-j_n+1} < \cdots < \sigma^{-1}_n)$ as

$$P_n^{\text{park}}(\sigma^{-1}_{n+1-j_n} < \sigma^{-1}_{n+1-j_n+1} < \cdots < \sigma^{-1}_n) = \sum_{\substack{m_r \geq 1; r=1,\cdots,j_n \\ \sum_{r=1}^{j_n} m_r \leq n}} P_n^{\text{park}}\left(n+1-\sigma^{-1}_{n-j_n+l} = \sum_{r=l}^{j_n} m_r,\ l=1,\cdots j_n\right).$$

Using this with (6.1), we have

(6.6)
$$P_n^{\text{park}}(\sigma^{-1}_{n+1-j_n} < \sigma^{-1}_{n+1-j_n+1} < \cdots < \sigma^{-1}_n) = \sum_{\substack{m_r \geq 1; r=1,\cdots,j_n \\ \sum_{r=1}^{j_n} m_r \leq n}} \left(\prod_{r=1}^{j_n} \frac{m_r^{m_r-1}e^{-m_r}}{m_r!}\right) A(n,j_n;m_1\cdots,m_{j_n})B(n,j_n;m_1,\cdots,m_{j_n}).$$

If $m_r \leq N_n$ for $r=1,\cdots,j_n$, then trivially, $\sum_{r=1}^{j_n} m_r \leq j_n N_n$. Thus, if we choose j_n and N_n such that

(6.7)
$$\lim_{n\to\infty} A(n,j_n;m_1\cdots,m_{j_n}) = 1,\ \text{uniformly over } \{1 \leq m_r \leq N_n; r=1,\cdots,j_n\};$$
$$\lim_{n\to\infty} B(n,j_n;m_1,\cdots,m_{j_n}) = 1,\ \text{uniformly over } \{1 \leq m_r \leq N_n; r=1,\cdots,j_n\},$$

and such that

$$j_n N_n \leq n;\ \text{and } j_n = o(N_n^{\frac{1}{2}}), \tag{6.8}$$

then it will follow from (6.5)-(6.7) that

$$\lim_{n\to\infty} P_n^{\text{park}}(\sigma^{-1}_{n+1-j_n} < \sigma^{-1}_{n+1-j_n+1} < \cdots < \sigma^{-1}_n) = 1. \tag{6.9}$$

In the calculations below, we always assume that $1 \leq m_r \leq N_n$, for $r=1,\cdots,j_n$. We first consider $A(n,j_n;m_1\cdots,m_{j_n})$ in (6.2). From Taylor's remainder formula, we have

$$-x \geq \log(1-x) \geq -x - \frac{1}{2}x^2\frac{1}{(1-x)^2},\ x \in (0,1). \tag{6.10}$$

Using (6.10), we have

$$\begin{aligned} &-\frac{n-1}{n+1}\sum_{r=1}^{j_n} m_r \geq \log\left(1-\frac{\sum_{r=1}^{j_n} m_r}{n+1}\right)^{n-1} \geq \\ &-\frac{n-1}{n+1}\sum_{r=1}^{j_n} m_r - (n-1)\left(\frac{1}{2}\frac{(\sum_{r=1}^{j_n} m_r)^2}{(n+1-\sum_{r=1}^{j_n} m_r)^2}\right). \end{aligned} \tag{6.11}$$

From (6.2) and (6.11), we have

$$\frac{2}{n+1}\sum_{r=1}^{j_n} m_r \geq \log A(n,j_n;m_1\cdots,m_{j_n}) \geq \tag{6.12}$$
$$\frac{2}{n+1}\sum_{r=1}^{j_n} m_r - \frac{(n-1)(\sum_{r=1}^{j_n} m_r)^2}{2(n+1-\sum_{r=1}^{j_n} m_r)^2}.$$

Recalling that $\sum_{r=1}^{j_n} m_r \leq j_n N_n$, it follows from (6.12) that

$$\text{(6.7) holds for } A(n,j_n;m_1\cdots,m_{j_n}) \text{ if } j_nN_n = o(n^{\frac{1}{2}}). \tag{6.13}$$

We now turn to $B(n,j_n;m_1\cdots,m_{j_n})$ in (6.2). We write

$$B(n,j_n;m_1\cdots,m_{j_n}) = C(n,j_n;m_1\cdots,m_{j_n})D(n,j_n;m_1\cdots,m_{j_n}), \tag{6.14}$$

where

$$C(n,j_n;m_1\cdots,m_{j_n}) = \frac{(n-j_n)!}{(n-\sum_{r=1}^{j_n} m_r)!}n^{j_n-\sum_{r=1}^{j_n} m_r}; \tag{6.15}$$
$$D(n,j_n;m_1\cdots,m_{j_n}) = \frac{\prod_{r=1}^{j_n}(n+1-\sum_{r=l}^{j_n} m_r)}{(n+1-\sum_{r=1}^{j_n} m_r)^{\sum_{r=1}^{j_n} m_r}}n^{\sum_{r=1}^{j_n} m_r - j_n}.$$

Let $w_n = \sum_{r=1}^{j_n} m_r$. We suppress the dependence of w_n on $\{m_r\}_{r=1}^{j_n}$. For any choice of $\{m_r\}_{r=1}^{j_n}$, we have $j_n \leq w_n \leq j_nN_n$.

We have

$$(n-w_n)^{w_n-j_n}n^{j_n-w_n} \leq C(n,j_n;m_1\cdots,m_{j_n}) \leq (n-j_n)^{w_n-j_n}n^{j_n-w_n},$$

or equivalently,

$$\Big(1-\frac{w_n}{n}\Big)^{w_n-j_n} \leq C(n,j_n;m_1\cdots,m_{j_n}) \leq \Big(1-\frac{j_n}{n}\Big)^{w_n-j_n}. \tag{6.16}$$

Since $w_n \leq j_nN_n$, it follows from (6.16) that
(6.17)
if $j_nN_n = o(n^{\frac{1}{2}}),$ then
$$\lim_{n\to\infty} C(n,j_n;m_1\cdots,m_{j_n}) = 1, \text{ uniformly over } \{1\leq m_r\leq N_n; r=1,\cdots,j_n\}.$$

We have

$$\frac{(n+1-w_n)^{j_n}}{(n+1-w_n)^{w_n}}n^{w_n-j_n} \leq D(n,j_n;m_1\cdots,m_{j_n}) \leq \frac{n^{j_n}}{(n+1-w_n)^{w_n}}n^{w_n-j_n},$$

or equivalently,

(6.18)
$$\left(1-\frac{w_n-1}{n}\right)^{j_n}\left(1+\frac{w_n-1}{n+1-w_n}\right)^{w_n}\le D_n\le\left(1+\frac{w_n-1}{n+1-w_n}\right)^{w_n}.$$

Since $w_n \le j_nN_n$, it follows from (6.18) that

(6.19)
if $j_nN_n = o(n^{\frac{1}{2}})$, then

$$\lim_{n\to\infty} D(n, j_n; m_1\cdots, m_{j_n}) = 1, \text{ uniformly over } \{1\le m_r\le N_n; r = 1,\cdots, j_n\}.$$

From (6.14), (6.17) and (6.19), it follows that

(6.20) (6.7) holds for $B(n, j_n; m_1\cdots, m_{j_n})$ if $j_nN_n = o(n^{\frac{1}{2}})$.

From (6.13), (6.20) and (6.8), we conclude that (6.9) holds if $j_nN_n = o(n^{\frac{1}{2}})$ and $j_n = o(N_n^{\frac{1}{2}})$. Thus, if $j_n = o(n^{\frac{1}{6}})$ and $N_n = [n^{\frac{1}{3}}]$, then (6.9) holds. This completes the proof of the theorem. □

7. Proof of Theorem 5

Part (ii) of the theorem follows immediately from part (i). For part (i), let i, j satisfy $1\le i\le j\le n$. We need to calculate the P_n^{park}-probability that $\sigma_i = j$ and that the location i is a left-to-right maximum for σ. To do this, we count how many parking functions $\{\pi_k\}_{k=1}^n$ yield a permutation $\sigma\in S_n$ with the above properties. A permutation $\sigma\in S_n$ satisfies the above properties if and only if $\{\sigma_1,\cdots,\sigma_{i-1}\}\subset[j-1]$ and $\sigma_i = j$. It follows from the definition of the parking process that a parking function $\{\pi_k\}_{k=1}^n$ yields such a permutation if and only if a subsequence of $\{\pi_1,\cdots,\pi_{j-1}\}$ of length $i-1$ constitutes a parking function of size $i-1$, the other $j-i$ elements of $\{\pi_1,\cdots,\pi_{j-1}\}$ belong to $[n]-[i]$, and $\pi_j\in[i]$. So we need to count how many such parking functions there are.

There are $\binom{j-1}{i-1}$ choices of $i-1$ elements for the subsequence of length $i-1$ from $\{\pi_1,\cdots,\pi_{j-1}\}$. There are i^{i-2} choices for such a subsequence of length $i-1$ to constitute a parking function of length $i-1$. There are i choices for π_j. So this gives $\binom{j-1}{i-1}i^{i-1}$ choices for π_j and for the $i-1$ elements of $\{\pi_1,\cdots,\pi_{j-1}\}$ that are restricted to $[i]$.

Denote by $\{\pi'_1,\cdots,\pi'_{j-i}\}$. the complementary subsequence of $j-i$ elements of $\{\pi_1,\cdots,\pi_{j-1}\}$. that all lie in $[n]-[i]$. The components $\{\pi_{j+1},\cdots,\pi_n\}$

are without specific restrictions. (But of course, $\{\pi'_1, \cdots, \pi'_{j-i}\}$ and $\{\pi_{j+1}, \cdots, \pi_n\}$ must be chosen so that $\{\pi_k\}_{k=1}^n$ is a parking function.)

In order to count how many choices there are for $\{\pi'_1, \cdots, \pi'_{j-i}\}$ and $\{\pi_{j+1}, \cdots, \pi_n\}$, we fix $l \in [n-j]$ and determine how many choices there are for $\{\pi'_1, \cdots, \pi'_{j-1}\}$ and $\{\pi_{j+1}, \cdots, \pi_n\}$, subject to the condition that a particular subsequence of $\{\pi_{j+1}, \cdots, \pi_n\}$ of length l has all of its elements in $[i]$, while the remaining elements in $\{\pi_{j+1}, \cdots, \pi_n\}$ are in $[n] - [i]$. Then we will sum over all such l. There are $\binom{n-j}{l}$ ways to choose the subsequence of length l from $\{\pi_{j+1}, \cdots, \pi_n\}$. There are i^l different possible values for these elements. Denote the subsequence of length $n-j-l$ of elements of $\{\pi_{j+1}, \cdots, \pi_n\}$ that are not in $[i]$ by $\{\pi'_{j+m}\}_{m=1}^{n-j-l}$.

We now count how many choices there are for $(\pi'_1, \cdots \pi'_{j-i}, \pi'_{j+1}, \cdots \pi'_{n-l})$. Our strategy will be to reduce the situation at hand to the following one (with appropriate choice of n and m): There is a one-way street with n spaces, but with the first m of them already taken up by a trailer. A sequence of $n-m$ cars enters, each with a preferred parking space between 1 and n. It was shown in [3] that the number of such sequences resulting in all $n-m$ cars successfully parking is $(m+1)(n+1)^{n-m-1}$.

By definition, all of the entries of the sequence $(\pi'_1, \cdots \pi'_{j-i}, \pi'_{j+1}, \cdots \pi'_{n-l})$ lie in $[n]-[i]$. Let $(\bar{\pi}'_1, \cdots, \bar{\pi}'_{n-i})$ denote the elements $(\pi'_1, \cdots, \pi'_{j-i}, \pi'_{j+1}, \cdots, \pi'_{n-l})$ along with the l fixed elements from $\{\pi_{j+1}, \cdots, \pi_n\}$ that are in $[i]$, using the order in which they appear in the parking function $\{\pi_k\}_{k=1}^n$. Counting the number of choices for $(\pi'_1, \cdots \pi'_{j-i}, \pi'_{j+1}, \cdots \pi'_{n-l})$ is equivalent to counting the number of choices for $(\bar{\pi}'_1, \cdots, \bar{\pi}'_{n-i})$, since the latter was obtained from the former by adding fixed components. Now let $(\pi''_1, \cdots, \pi''_{n-i})$ denote what one obtains if one starts with the sequence $(\bar{\pi}'_1, \cdots, \bar{\pi}'_{n-i})$ and slides the l entries that are in $[i]$ all the way to the left end of the sequence. So the first l entries in $(\pi''_1, \cdots, \pi''_{n-i})$ belong to $[i]$ and the $n-i-l$ other entries are $(\pi'_1, \cdots, \pi'_{j-i}, \pi'_{j+1}, \cdots, \pi'_{n-l})$, in the this order. By the symmetry of parking functions, counting the number of choices for $(\pi'_1, \cdots \pi'_{j-i}, \pi'_{j+1}, \cdots \pi'_{n-l})$ is equivalent to counting the number of choices for $(\pi''_1, \cdots, \pi''_{n-i})$.

We now count the number of choices for $(\pi''_1, \cdots, \pi''_{n-i})$. Remember that the first i parking spaces have already been filled. We need to consider how many choices there are for $(\pi''_1, \cdots, \pi''_{n-i})$ so that the remaining $n-i$

cars can park. By construction, the first l entries in $(\pi''_1, \cdots, \pi''_{n-i})$ involve no choice; they are fixed. Furthermore, all of these entries are in $[i]$. Thus, when using $(\pi''_1, \cdots, \pi''_{n-i})$, the first l cars from the $n-i$ cars will park in spaces $i+1, \cdots, i+l$. Now consider the state of affairs at this stage. The first $i+l$ positions are filled. What remains of the parking function is $(\pi''_{l+1}, \cdots, \pi''_{n-i})$, which has length $n-l-i$. All of the entries of $(\pi''_{l+1}, \cdots, \pi''_{n-i})$ lie in $[n]-[i]$. This is equivalent to the situation described above from [3]. Our case corresponds to the situation in [3] with m and n replaced respectively by l and $n-l-i$. More specifically, the number of choices for $(\pi''_{l+1}, \cdots, \pi''_{n-i})$ is the number of choices for a parking function $(\Pi_1, \cdots, \Pi_{n-l-i})$, all of whose entries have values in $[n-i]$ in the scenario where there are $n-i$ spaces and $n-i-l$ cars, and a trailer takes up the first l spaces. Thus, from the formula from [3] noted above, the number of such parking functions is $(l+1)(n-i+1)^{n-i-l-1}$. So we conclude that there are $(l+1)(n-i+1)^{n-i-l-1}$ choices for $(\pi'_1, \cdots \pi'_{j-i}, \pi'_{j+1}, \cdots \pi'_{n-l})$.

From the above analysis, we conclude that the number of parking functions $\{\pi_k\}_{k=1}^n$ that yield a permutation $\sigma \in S_n$ for which $\sigma_i = j$ and the location i is a left-to-right maximum is given by

$$\sum_{l=0}^{n-j} \binom{j-1}{i-1} i^{i-1} \binom{n-j}{l} i^l (l+1)(n-i+1)^{n-i-l-1}. \tag{7.1}$$

A standard calculation gives

$$\sum_{l=0}^{n-j} (l+1) \binom{n-j}{l} i^l (n-i+1)^{n-j-l} = i(n-j)(n+1)^{n-j-1} + (n+1)^{n-j}. \tag{7.2}$$

Thus, (7.1) is equal to

$$\binom{j-1}{i-1} i^{i-1} (n-i+1)^{j-i-1} \left(i(n-j)(n+1)^{n-j-1} + (n+1)^{n-j} \right).$$

Since there are $(n+1)^{n-1}$ parking functions $\{\pi_k\}_{k=1}^n$ of length n, it follows that

$$P_n^{\text{park}}(i \text{ is a left-to-right maximum and } \sigma_i = j) =$$
$$\binom{j-1}{i-1} i^{i-1} (n-i+1)^{j-i-1} \left(i(n-j)(n+1)^{-j} + (n+1)^{1-j} \right),$$

which is (1.24). □

8. Proof of part (i) of Theorem 6

We first proof (1.26). In light of Theorem 5, we need to show that the right hand side of (1.25) is equal to the right hand side of (1.26). We write the right hand side of (1.25) as

(8.1)
$$\sum_{j=1}^{n}\sum_{i=1}^{j}\binom{j-1}{i-1}i^{i-1}(n-i+1)^{j-i-1}\left(i(n-j)(n+1)^{-j}+(n+1)^{1-j}\right)=I+II,$$

where $I=\sum_{j=1}^{n}\sum_{i=1}^{j}\binom{j-1}{i-1}i^{i}(n-i+1)^{j-i-1}(n-j)(n+1)^{-j}$

and $II=\sum_{j=1}^{n}\sum_{i=1}^{j}\binom{j-1}{i-1}i^{i-1}(n-i+1)^{j-i-1}(n+1)^{1-j}$.

In [6], chapter 1 considers sums of the form

$$A_n(x,y;p,q):=\sum_{k=0}^{n}\binom{n}{k}(x+k)^{k+p}(y+n-k)^{n-k+q}.$$

When $p=-1$ and $q=0$, one has $A_n(x,y;-1,0)=\frac{(x+y+n)^n}{x}$, which is known as Abel's generalization of the binomial theorem. It is easy to see that the symmetry formula $A_n(x,y;p,q)=A_n(y,x;q,p)$ holds; thus,

$$A_n(x,y;0,-1)=\frac{(x+y+n)^n}{y}. \tag{8.2}$$

This allows us to calculate II. We have

$$\sum_{i=1}^{j}\binom{j-1}{i-1}i^{i-1}(n-i+1)^{j-i-1}=\sum_{r=0}^{j-1}\binom{j-1}{r}(r+1)^r(n-r)^{(j-1)-r-1}=$$
$$A_{j-1}(1,n-j+1;0,-1)=\frac{(n+1)^{j-1}}{n-j+1}.$$

Thus, we conclude that

$$II=\sum_{j=1}^{n}\frac{(n+1)^{j-1}}{n-j+1}(n+1)^{1-j}=\sum_{j=1}^{n}\frac{1}{j}. \tag{8.3}$$

We now turn to I. The sum over i that we now need to consider is

(8.4)
$$\sum_{i=1}^{j}\binom{j-1}{i-1}i^i(n-i+1)^{j-i-1}=\sum_{r=0}^{j-1}\binom{j-1}{r}(r+1)^{r+1}(n-r)^{(j-1)-r-1}=$$
$$A_{j-1}(1,n-j+1;1,-1).$$

In [6], a formula for $A_n(x,y;-1,1)$ $(=A_n(y,x;1,-1))$ is given, but we were unable to exploit it for our purposes. However, we were able to exploit the recursion formula

$$A_n(x,y;p,q)=A_{n-1}(x,y+1;p,q+1)+A_{n-1}(x+1,y;p+1,q),$$

which can also be found in [6]. Using this recursion formula, we have

$$A_{j-1}(1,n-j+1;1,-1)=A_j(0,n-j+1;0,-1)-A_{j-1}(0,n-j+2;0,0). \tag{8.5}$$

By (8.2),

$$A_j(0,n-j+1;0,-1)=\frac{(n+1)^j}{n-j+1}. \tag{8.6}$$

By [6, page 23],

$$A_n(x,y;0,0)=\sum_{k=0}^{n}\binom{n}{k}(x+y+n)^k(n-k)!=n!\sum_{k=0}^{n}\frac{(x+y+n)^k}{k!}.$$

Thus,

$$A_{j-1}(0,n-j+2;0,0)=(j-1)!\sum_{k=0}^{j-1}\frac{(n+1)^k}{k!}. \tag{8.7}$$

From (8.5)-(8.7), we obtain

$$A_{j-1}(1,n-j+1;1,-1)=\frac{(n+1)^j}{n-j+1}-(j-1)!\sum_{k=0}^{j-1}\frac{(n+1)^k}{k!}. \tag{8.8}$$

Now (8.1), (8.4) and (8.8) yield

$$\begin{aligned} I=&\sum_{j=1}^{n}A_{j-1}(1,n-j+1;1,-1)(n-j)(n+1)^{-j}=\\ &\sum_{j=1}^{n}\frac{n-j}{n-j+1}-\sum_{j=1}^{n}(n-j)(j-1)!\sum_{k=0}^{j-1}\frac{(n+1)^{k-j}}{k!}.\end{aligned} \tag{8.9}$$

From (8.3) and (8.9), we have

$$
\begin{aligned}
I+II = n-\sum_{j=1}^{n}(n-j)(j-1)!\sum_{k=0}^{j-1}\frac{(n+1)^{k-j}}{k!} = \\
n-n\sum_{j=1}^{n}(j-1)!\sum_{k=0}^{j-1}\frac{(n+1)^{k-j}}{k!}+\sum_{j=1}^{n}j!\sum_{k=0}^{j-1}\frac{(n+1)^{k-j}}{k!}.
\end{aligned} \tag{8.10}
$$

Using the well-known identity $\sum_{i=0}^{r}\binom{m+i}{i}=\binom{m+r+1}{r}$ in the third equality below, we have

$$
\begin{aligned}
&\sum_{j=1}^{n}(j-1)!\sum_{k=0}^{j-1}\frac{(n+1)^{k-j}}{k!}=\sum_{l=1}^{n}(\frac{1}{n+1})^{l}\sum_{k=0}^{n-l}\frac{1}{k!}(k+l-1)! = \\
&\sum_{l=1}^{n}(\frac{1}{n+1})^{l}(l-1)!\sum_{k=0}^{n-l}\binom{k+l-1}{k}=\sum_{l=1}^{n}(\frac{1}{n+1})^{l}(l-1)!\binom{n}{l} = \\
&\sum_{l=1}^{n}\frac{1}{l}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^{l}}.
\end{aligned} \tag{8.11}
$$

Similarly, we have

$$
\sum_{j=1}^{n}j!\sum_{k=0}^{j-1}\frac{(n+1)^{k-j}}{k!}=\sum_{l=1}^{n}(\frac{1}{n+1})^{l}l!\binom{n+1}{l+1}=\sum_{l=1}^{n}\frac{n+1}{l+1}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^{l}}. \tag{8.12}
$$

Using (8.11) and (8.12), we can write the last two terms on the right hand side of (8.10) as

$$
\begin{aligned}
&-n\sum_{j=1}^{n}(j-1)!\sum_{k=0}^{j-1}\frac{(n+1)^{k-j}}{k!}+\sum_{j=1}^{n}j!\sum_{k=0}^{j-1}\frac{(n+1)^{k-j}}{k!} = \\
&\sum_{j=1}^{n}\left(-\frac{n}{l}+\frac{n+1}{l+1}\right)\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^{l}}=-\sum_{l=1}^{n}\frac{n-l}{l(l+1)}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^{l}}.
\end{aligned} \tag{8.13}
$$

From (8.1), (8.10) and (8.13), it follows that the right hand side of (1.25) is equal to the right hand side of (1.26). □

9. Proof of part (ii) of Theorem 6

In light of (1.26), we need to analyze the asymptotic behavior of $\sum_{l=1}^{n}\frac{n-l}{l(l+1)}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^{l}}$. By Taylor's remainder theorem,

$$
-x-\frac{1}{2}(\frac{x}{1-x})^{2}\le\log(1-x)\le -x,\ 0\le x<1.
$$

Writing $\log \frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^l} = \log \prod_{k=1}^{l}(1-\frac{k}{n+1}) = \sum_{k=1}^{l}\log(1-\frac{k}{n+1})$, we then have

$$-\sum_{k=1}^{l}\left(\frac{k}{n+1}+\frac{1}{2}\left(\frac{k}{n+1-k}\right)^2\right) \le \log\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^l} \le -\sum_{k=1}^{l}\frac{k}{n+1},$$

and thus

$$e^{-\frac{l(l+1)}{2(n+1)}}e^{-\frac{l^3}{2(n+1-l)^2}} \le \frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^l} \le e^{-\frac{l(l+1)}{2(n+1)}}, \tag{9.1}$$

where we've used the fact that $\sum_{k=1}^{l}\frac{1}{2}\left(\frac{k}{n+1-k}\right)^2 \le \frac{l^3}{2(n+1-l)^2}$. In what follows, whenever we write that an expression depending on n is $\theta^+(n^\alpha)$, we mean that it falls between c_1n^α and c_2n^α for all n, where $c_1, c_2 > 0$.

We first prove the lower bound on $E_n^{\text{park}}X_n^{\text{LR-max}}$ in (1.27). Thus, we need an upper bound on $\sum_{l=1}^{n}\frac{n-l}{l(l+1)}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^l}$. We break up the sum into two parts. In turns out that the optimal intermediate point at which to break up the sum is $\theta^+(n^{\frac{1}{2}})$. Thus, using (9.1) for the first inequality below, and recalling that $\sum_{l=1}^{n}\frac{1}{l(l+1)} = \sum_{l=1}^{n}\left(\frac{1}{l}-\frac{1}{l+1}\right) = 1-\frac{1}{n+1}$, we write

(9.2)

$$\sum_{l=1}^{n}\frac{n-l}{l(l+1)}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^l} \le \sum_{l=1}^{n}\frac{n-l}{l(l+1)}e^{-\frac{l(l+1)}{2(n+1)}} =$$

$$\sum_{l=1}^{[n^{\frac{1}{2}}]}\frac{n-l}{l(l+1)}e^{-\frac{l(l+1)}{2(n+1)}} + \sum_{[n^{\frac{1}{2}}]+1}^{n}\frac{n-l}{l(l+1)}e^{-\frac{l(l+1)}{2(n+1)}} \le \sum_{n=1}^{[n^{\frac{1}{2}}]}\frac{n-l}{l(l+1)}+$$

$$e^{-\frac{n^2}{2(n+1)}}\sum_{[n^{\frac{1}{2}}]+1}^{n}\frac{n-l}{l(l+1)} = \sum_{n=1}^{n}\frac{n-l}{l(l+1)} - \left(1-e^{-\frac{n^2}{2(n+1)}}\right)\sum_{[n^{\frac{1}{2}}]+1}^{n}\frac{n-l}{l(l+1)} =$$

$$n(1-\frac{1}{n+1}) - \sum_{l=1}^{n}\frac{1}{l+1} - \theta^+(n^{\frac{1}{2}}) = n - \theta^+(n^{\frac{1}{2}}).$$

The lower bound in (1.27) now follows from (1.26) and (9.2).

We now turn to the proof of the upper bound in (1.27). Thus, now we need a lower bound on $\sum_{l=1}^{n}\frac{n-l}{l(l+1)}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^l}$. Note that for $l = l_n = o(n)$, the exponent on the left hand side of (**??**) is increasing in l and satisfies $\frac{l_n(l_n+1)}{2(n+1)} + \frac{l^3}{2(n+1-l)^2} = \theta^+(\frac{l_n^2}{n})$. Thus, for any $r \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^{r}$ satisfying

$a_1 < \alpha_2 < \cdots < \alpha_r < 1$, we have from (9.1)

(9.3)

$$\sum_{l=1}^{n}\frac{n-l}{l(l+1)}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^l} \ge$$

$$e^{-\theta^+\left(n^{2\alpha_1-1}\right)}\sum_{l=1}^{[n^{\alpha_1}]}\frac{n-l}{l(l+1)}+\sum_{i=2}^{r}e^{-\theta^+\left(n^{2\alpha_i-1}\right)}\sum_{l=[n^{\alpha_{i-1}}]+1}^{[n^{\alpha_i}]}\frac{n-l}{l(l+1)} \ge$$

$$\left(1-\theta^+\left(n^{2\alpha_1-1}\right)\right)\left(n(1-\frac{1}{[n^{\alpha_1}]+1})-\sum_{l=1}^{[n^{\alpha_1}]}\frac{1}{l+1}\right)+$$

$$\sum_{i=2}^{r}\left(1-\theta^+\left(n^{2\alpha_i-1}\right)\right)\left(n(\frac{1}{[n^{\alpha_{i-1}}]+1}-\frac{1}{[n^{\alpha_i}]+1})-\sum_{l=[n^{\alpha_{i-1}}]+1}^{[n^{\alpha_i}]}\frac{1}{l+1}\right).$$

Note that

(9.4)

$$\left(1-\theta^+\left(n^{2\alpha_1-1}\right)\right)\left(n(1-\frac{1}{[n^{\alpha_1}]+1})-\sum_{l=1}^{[n^{\alpha_1}]}\frac{1}{l+1}\right)=$$

$$n(1-\frac{1}{[n^{\alpha_1}]+1})-\theta^+\left(n^{2\alpha_1}\right);$$

$$\left(1-\theta^+\left(n^{2\alpha_i-1}\right)\right)\left(n(\frac{1}{[n^{\alpha_{i-1}}]+1}-\frac{1}{[n^{\alpha_i}]+1})-\sum_{l=[n^{\alpha_{i-1}}]+1}^{[n^{\alpha_i}]}\frac{1}{l+1}\right)=$$

$$n(\frac{1}{[n^{\alpha_{i-1}}]+1}-\frac{1}{[n^{\alpha_i}]+1})-\theta^+\left(n^{2\alpha_i-\alpha_{i-1}}\right),\ i=2,\cdots,r.$$

From (9.4) and (9.3), we obtain

$$\sum_{l=1}^{n}\frac{n-l}{l(l+1)}\frac{\prod_{i=0}^{l-1}(n-i)}{(n+1)^l} \ge \tag{9.5}$$

$$n-\theta^+\left(n^{1-\alpha_r}\right)-\theta^+\left(n^{2\alpha_1}\right)-\sum_{i=2}^{r}\theta^+\left(n^{2\alpha_i-\alpha_{i-1}}\right).$$

We now set $\{\alpha_i\}_{i=1}^r$ in order to minimize the order of $\theta^+\left(n^{1-\alpha_r}\right)+\theta^+\left(n^{2\alpha_1}\right)+\sum_{i=2}^{r}\theta^+\left(n^{2\alpha_i-\alpha_{i-1}}\right)$. The minimal will occur when all of the exponents $1-\alpha_r, 2\alpha_1, \{2\alpha_i-\alpha_{i-1}\}_{i=2}^r$ are equal to one another. Solving these equations, one obtains

$$\alpha_j=\frac{2^j-1}{2^{j-1}}\frac{2^{r-1}}{2^{r+1}-1},\ j=1,\cdots,r.$$

The common value of the exponents is then $\frac{2^r}{2^{r+1}-1}$ Therefore, from (1.26) and (9.5), we conclude that

$$E_n^{\text{park}} X_n^{\text{LR-max}} \le \theta^+ \left(n^{\frac{2^r}{2^{r+1}-1}} \right).$$

Since $r \in \mathbb{N}$ is arbitrary, this proves the upper bound in (1.27). □

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Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

Email address: pinsky@technion.ac.il

URL: https://pinsky.net.technion.ac.il/