TWO MEASURES OF EFFICIENCY FOR THE SECRETARY PROBLEM WITH MULTIPLE ITEMS AT EACH RANK

ROSS G. PINSKY

Abstract. For $2 \le k \in \mathbb{N}$, consider the following adaptation of the classical secretary problem. There are k items at each of n linearly ordered ranks. The kn items are revealed, one item at a time, in a uniformly random order, to an observer whose objective is to select an item of highest rank. At each stage the observer only knows the relative ranks of the items that have arrived thus far, and must either select the current item, in which case the process terminates, or reject it and continue to the next item. For $M \in \{0, 1, \cdots, kn-1\}$, let $\mathcal{S}(n,k;M)$ denote the strategy whereby one allows the first M items to pass, and then selects the first later arriving item whose rank is either equal to or greater than the highest rank of the first M items (if such an item exists). Let $W_{\mathcal{S}(n,k;M)}$ denote the event that one selects an item of highest rank using strategy $\mathcal{S}(n,k;M)$ and let $P_{n,k}(W_{\mathcal{S}(n,k;M)})$ denote the corresponding probability. We obtain a formula for $P_{n,k}(W_{\mathcal{S}(n,k;M)})$, and for $\lim_{n\to\infty} P_{n,k}(W_{\mathcal{S}(n,k;M_n)})$, when $M_n \sim ckn$, with $c \in (0,1)$. In the classical secretary problem ($k=1$), the asymptotic probability of success using an optimal strategy is $\frac{1}{e} \approx 0.368$. For $k=2$, the asymptotic probability of success using an optimal strategy from the above class is about 0.701. For $k=7$, that probability already exceeds 0.99. In the problem with multiple items at each rank, there is an additional measure of efficiency of a strategy besides the probability of selecting an item of highest rank; namely how quickly one selects an item of highest rank. We give a rather complete picture of this efficiency.

1. Introduction and Statement of Results

Recall the classical secretary problem: For $n \in \mathbb{N}$, a set of n linearly ranked items is revealed, one item at a time, to an observer whose objective is to select the item of highest rank. The order of the items is uniformly

2000 *Mathematics Subject Classification.* 60G40, 60C05.

Key words and phrases. secretary problem, optimal stopping, permutations.

random; that is, each of the $n!$ permutations of the ranks is equally likely. At each stage, the observer only knows the relative ranks of the items that have arrived thus far, and must either select the current item, in which case the process terminates, or reject it and continue to the next item. If the observer rejects the first $n-1$ items, then the nth and final item to arrive must be accepted. For $M \in \{0, 1, \cdots, n-1\}$, let $\mathcal{S}(n; M)$ denote the strategy whereby one lets the first M items pass, and then selects the first later arriving item that is ranked higher than all of the first M items (if such an item exists). As is very well known, asymptotically as $n \to \infty$, the optimal strategy is $\mathcal{S}(n; M_n)$, where $M_n \sim \frac{n}{e}$, and the limiting probability of successfully selecting the item of highest rank is $\frac{1}{e} \approx 0.368$.

In this paper, we extend the secretary problem to the case that there are multiple items at each rank. Fix an integer $2 \le k \in \mathbb{N}$. For $n \in \mathbb{N}$, consider n linearly ordered ranks and kn items, with k items at each rank. The kn items are revealed, one item at a time, to an observer whose objective is to select an item of highest rank. The order of the items is uniformly random. At each stage the observer only knows the relative ranks of the items that have arrived thus far, and must either select the current item, in which case the process terminates, or reject it and continue to the next item. Thus, the problem is equivalent to the problem of revealing the items one by one in a uniformly random permutation of the set $\cup_{i=1}^n \cup_{l=1}^k \{i_l\}$, which consists of k repetitions of each number $i \in [n]$. The permutations of this set will be denoted by $S_{n,k}$. (Of course, $S_{n,k}$ is equivalent to S_{nk}, the set of permutations of $[nk]$.) Each permutation in $S_{n,k}$ has nk positions. Here and in the sequel, the items are always revealed according to the left to right order of their positions in the permutation. The items $\{n_l\}_{l=1}^k$ are considered the items of highest rank.

In the classical secretary problem, there is only one measure of efficiency; namely the probability of selecting the item of highest rank. In the problem at hand, there is an additional measure of efficiency; namely how quickly one selects an item of highest rank. Indeed, as we shall see, this latter measure of efficiency becomes the dominant one because for all but the first few values of k, the class of strategies we consider is capable of selecting an item of highest rank with probability extremely close to one.

We consider a class of strategies of a form similar to, but slightly different from, the above mentioned strategies in the classical secretary problem. For $0 \leq M < nk$, denote by $\mathcal{S}(n, k; M)$ the strategy whereby one allows the first M items to pass, and then selects the first later arriving item whose rank is either equal to or greater than the highest rank of the first M items (if such an item exists).

The paper [5] considered a different and more intricate class of strategies for the secretary problem considered here, in the case that $k = 2$, and these strategies can in fact be defined for any k, as we now describe. For $0 \leq R \leq n$, let $\mathcal{T}(n, k; R)$ denote the strategy whereby one lets items pass until at least one item has occurred from R of the n different ranks. After that, one selects the first item to occur that satisfies the following two criteria: (1) $k - 1$ items of the same rank as the current item have already passed; and (2) the rank of the current item is larger than or equal to the rank of every item that has preceded it. If these two criteria never occur, then one fails.

It is intuitively clear that an optimal choice of strategies from the class $\mathcal{T}(n, k; R)$ will yield a higher probability of selecting an item of highest rank than will an optimal choice of strategies from the class $\mathcal{S}(n, k; M)$. However, as already noted, for all but the first few values of k, the optimal strategy from the class $\mathcal{S}(n, k; M)$ already yields probabilities extremely close to one. We will show that the class of strategies $\mathcal{S}(n, k; M)$ is overwhelmingly more efficient than the class $\mathcal{T}(n, k; R)$ from the point of view of how quickly an item of highest rank is selected.

Before presenting our precise results, we give a simple argument that shows that under an optimal strategy, the probability of selecting an item of highest rank is at least $1 - (\frac{1}{2})^{k-1}$, and thus very close to 1 for all but the first few values of k. For $c \in (0, 1)$, consider the strategy $\mathcal{S}(n, k; [ckn])$. That is, one let's the first $[ckn]$ items pass and then selects the first item whose rank is either equal to or greater than the highest rank of the first $[ckn]$ items. This strategy automatically produces an item of highest rank unless either all k items of highest rank happen to be among the first $[ckn]$ items, in which case it does not produce an item of highest rank, or none of the k items of highest rank happen to be among the first $[ckn]$ items, in which case it may or may not produce an item of highest rank. Denote by

A_n the event that all k items of highest rank are among the first $[ckn]$ items and by B_n the event that none of the k items of highest rank are among the first $[ckn]$ items. Then the probability of selecting an item of highest rank is at least $1-P(A_n)-P(B_n)$. We have

$$P(A_n)=\frac{\binom{k(n-1)}{[ckn]-k}}{\binom{kn}{[ckn]}};\quad P(B_n)=\frac{\binom{k(n-1)}{[ckn]}}{\binom{kn}{[ckn]}},$$

from which we obtain $\lim_{n\to\infty}P(A_n)=c^k$ and $\lim_{n\to\infty}P(B_n)=(1-c)^k$. Choosing $c=\frac12$, we conclude that the probability of selecting an item of highest rank is indeed at least $1-(\frac12)^{k-1}$.

We now turn to the precise results. Let $P_{n,k}$ denote the uniform probability measure on the set of permutations $S_{n,k}$. Let $W_{\mathcal{S}(n,k;M)}\subset S_{n,k}$ denote the event that an item of highest rank is selected when using the strategy $\mathcal{S}(n,k;M)$. Our first result gives an exact formula for $P_{n,k}(W_{\mathcal{S}(n,k;M)})$. The first sum on the right hand side of (1.1) below is the probability of $(A_n\cup B_n)^c$, where A_n and B_n are as in the previous paragraph. The second sum on the right hand side of (1.1) is the probability of $B_n\cap W_{\mathcal{S}(n,k;M)}$. This is calculated by considering the highest rank seen among the first M items and the number of times this rank appears among the first M items. We use the convention $\binom{b}{a}=0$ if $a>b$.

Proposition 1.

$$\begin{aligned}P_{n,k}(W_{\mathcal{S}(n,k;M)})=&\sum_{l=1}^{k-1}\frac{\binom{k}{l}\binom{(n-1)k}{M-l}}{\binom{kn}{M}}+\\ &k\sum_{j=1}^{n-1}\sum_{l=1}^{k}\frac{\binom{k}{l}\binom{(j-1)k}{M-l}}{\binom{kn}{M}}\frac{1}{k(n-j+1)-l}.\end{aligned}\tag{1.1}$$

An asymptotic analysis of the formula in Proposition 1 leads to the central result of this paper.

Theorem 1. *Let $M_n\sim ckn$, where $c\in(0,1)$. Then*

$$\begin{aligned}\lim_{n\to\infty}P_{n,k}(W_{\mathcal{S}(n,k;M_n)})=&-(1-c)^k\sum_{l=1}^{k-1}\binom{k}{l}(\frac{c}{1-c})^l\frac{l}{k-l}+\\ &k\sum_{l=1}^{k-1}\binom{k}{l}c^l\int_0^{1-c}\frac{y^{k-l-1}}{1-y^k}dy-c^k\log(1-(1-c)^k).\end{aligned}\tag{1.2}$$

Using partial fractions and trigonometric substitution, we can calculate explicitly the integrals on the right hand side above for the cases $k = 2, 3$. We obtain

(1.3)

$$\lim_{n\to\infty} P_{n,2}(W_{\mathcal{S}(n,2;M_n)}) = -2c(1-c) + (2c-c^2)\log(2-c) - (2c+c^2)\log c;$$

$$\lim_{n\to\infty} P_{n,3}(W_{\mathcal{S}(n,3;M_n)}) = -\frac{3}{2}(1-c)c(1+3c) - (3c+3c^2+c^3)\log c +$$

$$(\frac{3}{2}c + \frac{3}{2}c^2 - c^3)\log(c^2-3c+3) + 3\sqrt{3}(-c+c^2)\arctan(\frac{3-2c}{\sqrt{3}}) + \frac{\sqrt{3}\pi}{2}(c-c^2).$$

Table 1 gives the approximate optimal value of c and the corresponding optimal limiting probability of selecting an item of highest rank for k between 1 and 10. For $k = 2$, the probability of success is about .701, compared to $\frac{1}{e} \approx .368$ in the classical case when there is only one item of each rank, while the optimal choice of c, namely, $c \approx 0.386$, is close to the optimal value $\frac{1}{e}$ in the classical case. For $k = 3$, the optimal probability is above 0.85, for $k = 7$, that probability exceeds 0.99, and for $k = 10$, it is approximately 0.999.

For $k \geq 12$, the optimal probability of success is 1.000, when rounded off to three decimal places. Table 2 considers several such values of k and gives the approximate range of values of c for which the probability of selecting an item of highest rank is approximately equal to 1.000.

k	argmax for c	max. prob.
2	0.386	0.701
3	0.413	0.854
4	0.431	0.928
5	0.444	0.964
6	0.453	0.982
7	0.460	0.991
8	0.465	0.996
9	0.465	0.996
10	0.472	0.999

TABLE 1. Optimal c and optimal probability

k	range for c	probability
12	[.44,.52]	1.000
15	[.36,.60]	1.000
25	[.24,.73]	1.000
50	[.13,.85]	1.000

Table 2. Range of c for which probability ≈ 1.000

As noted above, the paper [5] considered the class of strategies $\mathcal{T}(n,k;R)$ in the case $k=2$. Let $W_{\mathcal{T}(n,k;R)} \subset S_{n,k}$ denote the event that an item of highest rank is selected when using the strategy $\mathcal{T}(n,k;R)$. It was shown there that the limiting probability of success, $\lim_{n\to\infty} P_{n,k}(W_{\mathcal{T}(n,2;R_n)})$, with $R_n \sim cn$, $c \in (0,1)$, is maximized for $c \approx 0.4709$, and the corresponding limiting probability is approximately 0.7680. This probability is higher than the probability 0.701 obtained using strategy $\mathcal{S}(n,2;M_n)$ with $M_n \sim 0.386(2n)$ as in Table 1. It isn't obvious how to extend the analysis in [5] to $k \geq 3$. On the other hand, in light of the results in Tables 1 and 2, for all but the first few values of k, a strategy from the class $\mathcal{T}(n,k;R)$ can hardly be more effective at selecting an item of highest rank than is the optimal strategy from the class $\mathcal{S}(n,k;M)$. Furthermore, we now show that the class of strategies $\mathcal{S}(n,k;M)$ has an overwhelming advantage over the class of strategies $\mathcal{T}(n,k;R)$. Indeed, note that if strategy $\mathcal{T}(n,k;R)$ succeeds, then it automatically selects the kth and final occurrence of an item with rank n, whereas strategy $\mathcal{S}(n,k;M_n)$ can succeed on an earlier occurrence of an item with rank n. Taking this point of view, we now investigate and compare the efficiency of these two classes of strategies.

On $W_{\mathcal{S}(n,k;M_n)} \subset S_{n,k}$, the event that an item of highest rank is selected when using strategy $\mathcal{S}(n,k;M_n)$, define $J^{(\mathcal{S})}_{n,k}$ with values in $[k]$ by $J^{(\mathcal{S})}_{n,k} = i$, if the item of rank n that was selected from among the k items $\{n_l\}_{l=1}^k$ of rank n, is the ith item of rank n to occur. Also, on the event $W_{\mathcal{S}(n,k;M_n)}$, let $X^{(\mathcal{S})}_{n,k}$ denote the position in $[kn]$ in which the selected item of highest rank occurs. (The dependence on $\{M_n\}$ in $J^{(\mathcal{S})}_{n,k}$ and $X^{(\mathcal{S})}_{n,k}$ is implicit in $\mathcal{S} = \mathcal{S}(n,k;M_n)$.)

On $W_{\mathcal{T}(n,k;R_n)} \subset S(n,k)$, the event that an item of highest rank is selected when using strategy $\mathcal{T}(n,k;R_n)$, define $J^{(\mathcal{T})}_{n,k} = k$, because from the definition of strategy $\mathcal{T}$, the item of highest rank n that was selected is automatically the kth and final item of rank n to occur. Also, on the event $W_{\mathcal{T}(n,k;R_n)}$, let $X^{(\mathcal{T})}_{n,k} \in [kn]$ denote the position in which the selected item of highest rank occurs. (The dependence on $\{T_n\}$ in $X^{(\mathcal{T})}_{n,k}$ is implicit in $\mathcal{T} = \mathcal{T}(n,k;R_n)$.)

Theorem 2. *Using the strategy $\mathcal{S}(n,k;M_n)$, where $M_n \sim ckn$, define*

$$p^{(k)}_c(i) = \lim_{n\to\infty} P_{n,k}(J^{(\mathcal{S})}_{n,k} = i|W_{\mathcal{S}(n,k;M_n)}),\ i \in [k].$$

That is, $p^{(k)}_c(i)$ is the limiting probability as $n \to \infty$ that the selected item of rank n is the ith item of rank n to occur. Then

(1.4)

$$p^{(k)}_c(i) = \begin{cases} \frac{1}{\lim_{n\to\infty} P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}(1-c)^{k-i+1}c^{i-1}\binom{k}{i-1},\ i \in \{2,\cdots,k\}; \\ 1 - \frac{1}{\lim_{n\to\infty} P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}\big(1-c^k-(1-c)^k\big),\ i=1. \end{cases}$$

Remark. In particular, if $\lim_{n\to\infty} P_{n,k}(W_{\mathcal{S}(n,k;M_n)}) \approx 1$ with $M_n \sim c(kn)$ (see Table 1), then

$$p^{(k)}_c(i) \approx \begin{cases} (1-c)^{k-i+1}c^{i-1}\binom{k}{i-1},\ i \in \{2,\cdots,k\}; \\ c^k + (1-c)^k,\ i=1. \end{cases}$$

Equivalently, the distribution $\{p^{(k)}_c(i)\}_{i=1}^k$ is approximately the distribution of the random variable $1 + X_{k,c}1_{X_{k,c}\neq k}$, where $X_{k,c}$ has the binomial distribution with parameters k and c.

Since $P_{n,k}(J^{(\mathcal{T})}_{n,k} = k|W_{\mathcal{T}(n,k;R_n)}) = 1$, it follows that in contrast to the above distribution $\{p^{(k)}_c(i)\}_{i=1}^k$ on $[k]$ corresponding to the class of strategies $\mathcal{S}(n,k;M)$, the analogous distribution on $[k]$ corresponding to the class of strategies $\mathcal{T}(n,k;R)$ is the degenerate distribution that places all its probability on $k \in [k]$.

The following theorem analyzes the quantity $\lim_{n\to\infty} \frac{E_{n,p}(X^{(\mathcal{S})}_{n,k}|W_{\mathcal{S}(n,k;M_n)})}{kn}$, the limiting expected proportion of items that are observed until an item of highest rank n is selected, conditioned on an item of highest rank begin selected, when using the strategy $\mathcal{S}(n,k;M_n)$.

Theorem 3. *Using the strategy $\mathcal{S}(n,k;M_n)$, where $M_n \sim ckn$, define*

$$\gamma^{(k)}(c) = \lim_{n\to\infty} \frac{E_{n,p}(X^{(\mathcal{S})}_{n,k}|W_{\mathcal{S}(n,k;M_n)})}{kn}.$$

That is, $\gamma^{(k)}(c)$ is the limiting expected proportion of items that are observed until an item of rank n is selected. Then

$$\gamma^{(k)}(c)=c+(1-c)\sum_{i=1}^{k}\frac{p_c^{(k)}(i)}{k+2-i}=$$

$$c+\frac{1-c}{\lim_{n\to\infty}P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}E\frac{1}{k+1-X_{k,c}}1_{X_{k,c}\not\in\{0,k\}}+ \tag{1.5}$$

$$\frac{1-c}{k+1}\big(1-\frac{1}{\lim_{n\to\infty}P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}(1-c^k-(1-c)^k)\big),$$

where $p_c^{(k)}(l)$ is as in (1.4) and $X_{k,c}$ is a binomial random variable with parameters k and c.

Remark. In cases where $\lim_{n\to\infty}P_{n,k}(W_{\mathcal{S}(n,k;M_n)})\approx 1$, (1.5) reduces to

$$\gamma^{(k)}(c)\approx c+(1-c)\sum_{i=1}^{k}(1-c)^{k-i+1}c^{i-1}\binom{k}{i-1}\frac{1}{k+2-i}+\frac{(1-c)c^k}{k+1}= \tag{1.6}$$

$$c+(1-c)E\frac{1}{k+1-X_{k,c}1_{X_{k,c}\neq k}}.$$

By an elementary large deviations result, $P(X_{k,c}\le(c-\epsilon)k)$ is exponentially small in k, for any $\epsilon>0$. From this, we obtain immediately the following corollary of (1.5).

Corollary 1. *For any sequence $\{c_k\}_{k=2}^\infty$ for which*

$$\liminf_{k\to\infty}\lim_{n\to\infty}P_{n,k}(W_{\mathcal{S}(n,k;M_n)})|_{M_n\sim c_k kn})>0$$

and $\limsup_{k\to\infty}c_k<1$, one has

$$\gamma^{(k)}(c_k)=c_k+\theta(\frac{1}{k}),\ \text{as } k\to\infty.$$

Table 3 gives the value of $\gamma^{k)}(c)$, the limiting expected proportion of items that are observed until an item of highest rank is selected, conditioned on an item of highest rank being selected, for $2\le k\le 10$, with the optimal value of c from Table 1. Table 4 considers several values of k with $k\ge 12$, the range of k for which the optimal probability is 1.000, to three decimal places. Numerical results show that the function in (1.6) is increasing in c. Thus, values are given for $\gamma^{(k)}(c)$ using the lowest value of c, taken from Table 2, for which the probability of success is approximately 1.000.

k	argmax for c from Table 1	$\gamma^{(k)}(c)$
2	0.386	0.660
3	0.413	0.636
4	0.431	0.618
5	0.444	0.604
6	0.453	0.592
7	0.460	0.583
8	0.465	0.575
9	0.465	0.565
10	0.472	0.563

TABLE 3. Value of $\gamma^{(k)}(c)$, the limiting expected proportion of items that are observed until an item of rank n is selected, at optimal c from Table 1

k	lower end approx. argmax for c	$\gamma^{(k)}(c)$
12	.44	.52
15	.36	.42
25	.24	.28
50	.13	.15

TABLE 4. Value of $\gamma^{(k)}(c)$, the limiting expected proportion of items that are observed until an item of rank n is selected, for the smallest value of c for which probability ≈ 1.000

With regard to the limiting expected proportion of items that are observed until an item of rank n is selected, in contrast to Theorem 3 and Tables 3 and 4 for the class of strategies $\mathcal{S}(n,k;M)$, we have the following result for the class of strategies $\mathcal{T}(n,k;R)$.

Proposition 2. *Using any sequence of strategies* $\{\mathcal{T}(n,k;R_n)\}$ *for which* $\alpha_k := \alpha_k(\{R_n\}) = \lim_{n\to\infty} P_{n,k}(W_{\mathcal{T}(n,k;R_n)})$ *exists, one has*

$$\begin{aligned}\frac{1}{\alpha_k}\frac{k}{k+1} - \frac{1-\alpha_k}{\alpha_k} \le \liminf_{n\to\infty} E_{n,k}(X_{n,k}^{(\mathcal{T})}|W_{\mathcal{T}(n,k;R_n)}) \le \\ \limsup_{n\to\infty} E_{n,k}(X_{n,k}^{(\mathcal{T})}|W_{\mathcal{T}(n,k;R_n)}) \le \frac{1}{\alpha_k}\frac{k}{k+1}.\end{aligned} \tag{1.7}$$

Remark. In particular, it follows that if $\lim_{n\to\infty} P_{n,k}(W_{\mathcal{T}(n,k;R_n)}) \approx 1$, then $\lim_{n\to\infty} E_{n,k}(X^{(\mathcal{T})}_{n,k}|W_{\mathcal{T}(n,k;R_n)}) \approx \frac{k}{k+1}$.

The classical secretary problem, where there is one item at each rank, but with adaptations to increase the chance of winning, go all the way back to the fundamental paper of Gilbert and Mosteller [6]. In particular, they considered the situation where one is given r opportunities to select the highest ranked item, as well as the situation in which one is given one opportunity to select an item from among the r top ranked items. In the first situation above, when there are r opportunities to select the highest ranked item, they showed that the asymptotic probability of winning when using the best strategy is about 0.591 for $r = 2$, and increases to about 0.965 when $r = 8$. In the second situation above, when there is one opportunity to select an item from among the r top ranked items, they showed that the asymptotic probability of winning when using the best strategy is about 0.574 for $r = 2$. The authors did not analyze their formula numerically in cases with $r > 2$. That paper also considered the so-called full information problem, where the "items" are now numbers obtained from a known IID sequence of continuous random variables. In this case, the optimal strategy yields a probability of about 0.58 for winning.

See [1] for another useful approach to the secretary problem. For the secretary problem in its classical setup, but with items arriving in a non-uniform order, see for example [6, 9, 10, 11]. The paper [2] considered the best strategy in order to minimize the expected rank of the selected item, using the convention that the lowest rank is best. The paper [12] considered the secretary problem when the number of items is random; see also, [7] and [8]. See [4] and [3] for reviews of the history of and a variety of variations on the secretary problem.

We prove Proposition 1 in section 2 and Theorem 1 in section 3. In section 4, we prove a couple of auxiliary results that are needed for the proofs of Proposition 2 and Theorem 3, and then give the proof of Proposition 2. In section 5 we prove Theorem 2 and in section 6 we prove Theorem 3. The proofs of these two theorems rely heavily on the calculations in the proofs of Proposition 1 and Theorem 1.

2. Proof of Proposition 1

Fix $2 \le k \in \mathbb{N}$. For $n \in \mathbb{N}$, we consider a uniformly random permutation of the multi-set $\{1^k, 2^k, \cdots, n^k\}$. The kn items of this multi-set arrive according to the order of this permutation. We consider n to be the highest rank and 1 to be the lowest rank. We wish to calculate the probability $P_{n,k}(W_{\mathcal{S}(n,k;M)})$ of winning under the strategy $\mathcal{S}(n,k;M)$.

Let $A^{(n)}_{M,j,l}$ denote the event that among the first M items, the number j has occurred l times, and no number greater than j has occurred at all, where $1 \le M \le kn-1,\ 1 \le j \le n$ and $l \in [k]$. To calculate $P_{n,k}(A^{(n)}_{M,j,l})$, it will be convenient to consider all nk objects as distinguishable from one another. (For each $i \in [n]$, think of the k different i's as $\{i_l\}_{l=1}^k$.) It then follows that $P_{n,k}(A^{(n)}_{M,j,l})$ is the following hypergeometric-type probability:

$$P_{n,k}(A^{(n)}_{M,j,l}) = \frac{\binom{k}{l}\binom{(j-1)k}{M-l}}{\binom{kn}{M}}, \tag{2.1}$$

where the above probability is considered to be zero if $M-l > (j-1)k$.

Consider first the case that $j \in [n-1]$. If one employs the strategy $\mathcal{S}(n,k;M)$, and the event $A^{(n)}_{M,j,l}$ occurs, then after the Mth item arrives, one will select the first item larger or equal to j. Among the items arriving after the Mth item arrives, there are $k(n-j+1)-l$ items that are larger or equal to j, of which k of them have the highest rank n. Since items arrive in uniformly random order,

$$P_{n,k}(W_{\mathcal{S}(n,k;M)}|A^{(n)}_{M,j,l}) = \frac{k}{k(n-j+1)-l},\ j \in [n-1]. \tag{2.2}$$

Consider now the case $j = n$. If one employs the strategy $\mathcal{S}(n,k;M)$, and the event $A^{(n)}_{M,n,l}$ occurs, then after the Mth item arrives, one will select the first item that is equal to n, if such an item exists. Of course, it will exist if $l \in [k-1]$ and it won't exist if $l = k$. Thus,

$$P_{n,k}(W_{\mathcal{S}(n,k;M)}|A^{(n)}_{M,n,l}) = \begin{cases} 1, & \text{if } l \in [k-1]; \\ 0, & \text{if } l = k. \end{cases} \tag{2.3}$$

Since for each M, the collection of events $\{A^{(n)}_{M,j,l} : j \in [n], l \in [k]\}$ are disjoint, and the probability of their union is 1, it follows from (2.1)-(2.3) that (1.1) holds. □

3. Proof of Theorem 1

It will be convenient to write the expression $\frac{\binom{k}{l}\binom{(j-1)k}{M-l}}{\binom{kn}{M}}$ appearing in (1.1) as

$$\frac{\binom{k}{l}\binom{(j-1)k}{M-l}}{\binom{kn}{M}}=\frac{\binom{M}{l}(k)_l\,(k(j-1))_{M-l}}{(kn)_M},\ j\in\{1,2,\cdots,n\},\tag{3.1}$$

where we use the notation $(b)_a=b(b-1)\cdots(b-a+1)$ for falling factorials, where $a,b\in\mathbb{Z}$.

Let $M_n=c_nkn$, be an integer for each $n\in\mathbb{N}$, with $\lim_{n\to\infty}c_n=c$. For $N\in\mathbb{N}$ and $M=M_n$, we use (3.1) and rewrite the second term on the right hand side of (1.1) as

$$\begin{aligned}&k\sum_{j=1}^{n-1}\sum_{l=1}^{k}\frac{\binom{M_n}{l}(k)_l(k(j-1))_{M_n-l}}{(kn)_{M_n}}\frac{1}{k(n-j+1)-l}=\\ &k\sum_{l=1}^{k}\binom{c_nkn}{l}(k)_l\sum_{j=1}^{n-N}\frac{(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{1}{k(n-j+1)-l}+\\ &k\sum_{l=1}^{k}\binom{c_nkn}{l}(k)_l\sum_{j=n-N+1}^{n-1}\frac{(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{1}{k(n-j+1)-l}.\end{aligned}\tag{3.2}$$

For sufficiently large n, there exists a constant $C=C(k,l,c)$ such that

$$\binom{c_nkn}{l}\frac{(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\le C\frac{(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn-l}}\le C\frac{(kj)_{c_nkn-l}}{(kn)_{c_nkn-l}}.\tag{3.3}$$

Using the fact that $\frac{a-x}{b-x}$ is decreasing in x for $0<x<a<b$, it follows that for any $c'\in(0,c)$ and for sufficiently large n,

$$\frac{(kj)_{c_nkn-l}}{(kn)_{c_nkn-l}}\le(\frac{kj}{kn})^{c_nkn-l}\le(\frac{j}{n})^{c'kn},\ l\in[k].\tag{3.4}$$

From (3.3) and (3.4) we have for large n,

$$\binom{c_nkn}{l}\sum_{j=1}^{n-N}\frac{(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{1}{k(n-j+1)-l}\le C\sum_{j=1}^{n-N}(\frac{j}{n})^{c'kn}.\tag{3.5}$$

Also,

$$\begin{aligned}&\sum_{j=1}^{n-N}(\frac{j}{n})^{c'kn}=n\sum_{j=1}^{n-N}\frac{1}{n}(\frac{j}{n})^{c'kn}\le n\int_0^{1-\frac{N-1}{n}}x^{c'kn}dx\le\\ &\frac{n}{c'kn+1}(1-\frac{N-1}{n})^{c'kn+1}\le\frac{n}{c'kn+1}e^{-\frac{N-1}{n}(c'kn+1)}.\end{aligned}\tag{3.6}$$

From (3.5) and (3.6), we conclude that the expression on the second line of (3.2) satisfies

(3.7)
$$\lim_{N\to\infty}\limsup_{n\to\infty} k\sum_{l=1}^{k}\binom{c_nkn}{l}(k)_l\sum_{j=1}^{n-N}\frac{(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{1}{k(n-j+1)-l}=0.$$

Letting $s=n-j$, we write the expression on the third line of (3.2) as

$$\begin{aligned}&k\sum_{l=1}^{k}\binom{c_nkn}{l}(k)_l\sum_{j=n-N+1}^{n-1}\frac{(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{1}{k(n-j+1)-l}=\\&k\sum_{l=1}^{k}\binom{c_nkn}{l}(k)_l\sum_{s=1}^{N-1}\frac{(k(n-s-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{1}{k(s+1)-l}.\end{aligned}\tag{3.8}$$

We write

(3.9)
$$\binom{c_nkn}{l}\frac{(k(n-s-1))_{c_nkn-l}}{(kn)_{c_nkn}}=\frac{\binom{c_nkn}{l}}{\prod_{i=1}^{l}(kn-c_nkn+i)}\prod_{t=0}^{c_nkn-l-1}\frac{k(n-s-1)-t}{kn-t}.$$

We have

$$\prod_{t=0}^{c_nkn-l-1}\frac{k(n-s-1)-t}{kn-t}=\prod_{i=(1-c_n)kn+l+1}^{kn}(1-\frac{k(s+1)}{i}),\tag{3.10}$$

and

(3.11)
$$\log\prod_{i=(1-c_n)kn+l+1}^{kn}(1-\frac{k(s+1)}{i})=\sum_{i=(1-c_n)kn+l+1}^{kn}\log(1-\frac{k(s+1)}{i})=$$
$$-k(s+1)\sum_{i=(1-c_n)kn+l+1}^{kn}\frac{1}{i}+o(1)=k(s+1)\log(1-c_n)+o(1),\ \text{as } n\to\infty.$$

From (3.10) and (3.11), we obtain

$$\prod_{t=0}^{c_nkn-l-1}\frac{k(n-s-1)-t}{kn-t}=(1+o(1))(1-c)^{k(s+1)},\ \text{as } n\to\infty.\tag{3.12}$$

Also, we have

$$\lim_{n\to\infty}\frac{\binom{c_nkn}{l}}{\prod_{i=1}^{l}(kn-c_nkn+i)}=\frac{1}{l!}(\frac{c}{1-c})^l.\tag{3.13}$$

Now (3.9), (3.12) and (3.13) yield

$$\lim_{n\to\infty}\binom{c_nkn}{l}\frac{(k(n-s-1))_{c_nkn-l}}{(kn)_{c_nkn}}=\frac{1}{l!}(\frac{c}{1-c})^l(1-c)^{k(s+1)}. \tag{3.14}$$

Letting $n\to\infty$ in (3.8), it follows from (3.14) that the expression on the third line of (3.2) satisfies

$$\begin{aligned}&\lim_{n\to\infty}k\sum_{l=1}^k\binom{c_nkn}{l}(k)_l\sum_{j=n-N+1}^{n-1}\frac{(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{1}{k(n-j+1)-l}=\\&k\sum_{l=1}^k\binom{k}{l}(\frac{c}{1-c})^l\sum_{s=1}^{N-1}(1-c)^{k(s+1)}\frac{1}{k(s+1)-l}.\end{aligned} \tag{3.15}$$

From (3.2),(3.7) and (3.15), we conclude that the second term on the right hand side of (1.1), with $M=M_n$, satisfies

(3.16)

$$\begin{aligned}&\lim_{n\to\infty}k\sum_{j=1}^{n-1}\sum_{l=1}^k\frac{\binom{M_n}{l}(k)_l(k(j-1))_{M_n-l}}{(kn)_{M_n}}\frac{1}{k(n-j+1)-l}=\\&k\sum_{l=1}^k\binom{k}{l}(\frac{c}{1-c})^l\sum_{s=1}^{\infty}(1-c)^{k(s+1)}\frac{1}{k(s+1)-l},\ \text{for } M_n\sim cn,\ c\in(0,1).\end{aligned}$$

We now consider the first term on the right hand side of (1.1), with $M=M_n=c_nkn\sim ckn$. Although we considered (3.14) for $s\ge 1$, of course it also holds for $s=0$. Thus, the first term on the right hand side of (1.1) satisfies

$$\lim_{n\to\infty}\sum_{l=1}^{k-1}\frac{\binom{M_n}{l}(k)_l(k(n-1))_{M_n-l}}{(kn)_{M_n}}=(1-c)^k\sum_{l=1}^{k-1}\binom{k}{l}(\frac{c}{1-c})^l. \tag{3.17}$$

From (1.1), (3.16) and (3.17), we obtain

(3.18)

$$\begin{aligned}&\lim_{n\to\infty}P_{n,k}(\mathcal{S}(n,k;M_n))=k\sum_{l=1}^k\binom{k}{l}(\frac{c}{1-c})^l\sum_{s=1}^{\infty}(1-c)^{k(s+1)}\frac{1}{k(s+1)-l}+\\&(1-c)^k\sum_{l=1}^{k-1}\binom{k}{l}(\frac{c}{1-c})^l,\ \text{for } M_n\sim cn,\ c\in(0,1).\end{aligned}$$

We now analyze the infinite series on the right hand side of (3.18).

Let

$$G_{k,l}(x)=\sum_{s=1}^{\infty}\frac{x^{k(s+1)}}{k(s+1)-l}. \tag{3.19}$$

Then

$$(x^{-l}G_{k,l})'=\sum_{s=1}^{\infty}x^{k(s+1)-l-1}=\frac{x^{2k-l-1}}{1-x^k}=\begin{cases}-x^{k-l-1}+\frac{x^{k-l-1}}{1-x^k},\ l=1,\cdots,k-1;\\ \frac{x^{k-1}}{1-x^k},\ l=k.\end{cases}$$

Integrating and noting that $G_{k,l}$ vanishes at zero to the order $2k$, we obtain

$$G_{k,l}(x)=\begin{cases}-\frac{x^k}{k-l}+x^l\int_0^x\frac{y^{k-l-1}}{1-y^k}dy,\ l=1,\cdots,k-1;\\ -\frac{x^k}{k}\log(1-x^k),\ l=k.\end{cases}\tag{3.20}$$

From (3.19) and (3.20), we can rewrite the first term on the right hand side of (3.18) as

$$\begin{aligned}&k\sum_{l=1}^{k}\binom{k}{l}(\frac{c}{1-c})^l\sum_{s=1}^{\infty}(1-c)^{k(s+1)}\frac{1}{(s+1)k-l}=\\ &-k(1-c)^k\sum_{l=1}^{k-1}\binom{k}{l}(\frac{c}{1-c})^l\frac{1}{k-l}+k\sum_{l=1}^{k-1}\binom{k}{l}c^l\int_0^{1-c}\frac{y^{k-l-1}}{1-y^k}dy-\\ &c^k\log(1-(1-c)^k).\end{aligned}\tag{3.21}$$

Now (1.2) follows from (3.18) and (3.21). □

4. Two auxiliary results and the proof of Proposition 2.

We state and prove two auxiliary results. The first one will be used in the proof of the second one and in the proof of Proposition 2, and the second one will be used in the proof of Theorem 3. We end this section with the proof of Proposition 2.

On the probability space $(S_{n,k},P_{n,k})$, for each $i\in[k]$, let $Y_i^{(n,k)}$ denote the position in $[kn]$ in which occurs the ith item to occur from among the k items $\{n_l\}_{l=1}^k$ of rank n. (For example, if $n=4$ and $k=2$, and the permutation in $S_{4,2}$ is given by $2_14_23_23_11_12_24_11_2$, then on this permutation, $Y_1^{(4,2)}$ takes the value 2 and $Y_2^{(4,2)}$ takes the value 7.)

Proposition 3. *For $k\in\mathcal{N}$ and $i\in[k]$,*

$$\lim_{n\to\infty}\frac{E_{n,k}Y_i^{(n,k)}}{nk}=\frac{i}{k+1}.\tag{4.1}$$

Proof. Since we will only need the cases $i=1$ and $i=k$, and since the latter case follows from the former one by considerations of symmetry, we will only proof the case $i=1$. The other cases can be proved similarly. The random variable $\frac{Y_1^{(n,k)}}{nk}$ on the probability space $(S_{n,k}, P_{n,k})$ is a discretization of the random variable $\min(U_1, \cdots, U_k)$, where $\{U_j\}_{j=1}^k$ are IID uniform random variables on $[0,1]$. Thus

$$\lim_{n\to\infty} \frac{E_{n,k}Y_1^{(n,k)}}{nk} = E\min(U_1, \cdots, U_k).$$

Since $P(\min(U_1, \cdots, U_k) \geq x) = (1-x)^k$, the density of $\min(U_1, \cdots, U_k)$ is $k(1-x)^{k-1}$. Thus,

$$E\min(U_1, \cdots, U_k) = \int_0^1 x\big(k(1-x)^{k-1}\big)dx = k\int_0^1 (1-x)x^{k-1}dx =$$
$$k(\frac{1}{k} - \frac{1}{k+1}) = \frac{1}{k+1}.$$

□

Proposition 4. *A box contains N black balls and A red balls. Draw the balls from the box without replacement until a red ball is obtained, and let $\tau^{(N,A)}$ denote the number of draws required. Then*

$$\lim_{N\to\infty} \frac{E\tau^{(N,A)}}{N} = \frac{1}{A+1}. \tag{4.2}$$

Proof. Consider drawing all of the balls from the box, where the black balls are numbered from 1 to N and the red balls are numbered from 1 to A. Every particular order in which the balls are drawn is a permutation of the $N+A$ items. Thus, if N is a multiple of A, say $N = MA$, for $M \in \mathbb{N}$, then $\tau^{(N,A)}$ is equivalent to $Y_1^{(n,k)}$ in Proposition 4.1, with $n = M+1$ and $k = A$. Therefore, from Proposition 4.1, $\lim_{M\to\infty} \frac{E\tau^{(MA,A)}}{(M+1)A} = \frac{1}{A+1}$, or equivalently,

$$\lim_{M\to\infty} \frac{E\tau^{(MA,A)}}{MA} = \frac{1}{A+1}. \tag{4.3}$$

A simple argument extends (4.3) to the case that MA is replaced by a generic N. □

Proof of Proposition 2. As in the statement of the proposition, let $\alpha_k = \lim_{n\to\infty} P_{n,k}(W_{\mathcal{T}(n,k;R_n)})$. From the definition of the strategy $\mathcal{T}(n,k;R_n)$, it follows that $X_{n,k}^{(\mathcal{T})} = Y_k^{(n,k)}|_{W_{\mathcal{T}(n,k;R_n)}}$ where $Y_k^{(n,k)}$ is as defined before Proposition 3. (Recall that $X_{n,k}^{(\mathcal{T})}$ is only defined on $W_{\mathcal{T}(n,k;R_n)}$.) Thus,

$$E_{n,k}(X_{n,k}^{(\mathcal{T})}|W_{\mathcal{T}(n,k;R_n)}) = E_{n,k}(Y_k^{(n,k)}|W_{\mathcal{T}(n,k;R_n)}) \leq \frac{E_{n,k}Y_k^{(n,k)}}{P_{n,k}(W_{\mathcal{T}(n,k;R_n)})}, \tag{4.4}$$

and then from (4.4) and (4.1)

$$\limsup_{n\to\infty}\frac{1}{nk}E_{n,k}(X^{(\mathcal{T})}_{n,k}|W_{\mathcal{T}(n,k;R_n)})\le\frac{1}{\alpha_k}\frac{k}{k+1}. \tag{4.5}$$

Using the notation $E(Z;B):=EZ1_B$, where Z is a random variable and B is an event, we also have

(4.6)

$$E_{n,k}(X^{(\mathcal{T})}_{n,k}|W_{\mathcal{T}(n,k;R_n)})=E_{n,k}(Y^{(n,k)}_k|W_{\mathcal{T}(n,k;R_n)})=\frac{E_{n,k}(Y^{(n,k)}_k;W_{\mathcal{T}(n,k;R_n)})}{P_{n,k}(W_{\mathcal{T}(n,k;R_n)})}\ge$$

$$\frac{E_{n,k}Y^{(n,k)}_k}{P_{n,k}(W_{\mathcal{T}(n,k;R_n)})}-nk\frac{1-P_{n,k}(W_{\mathcal{T}(n,k;R_n)})}{P_{n,k}(W_{\mathcal{T}(n,k;R_n)})},$$

and then from (4.6) and (4.1),

$$\liminf_{n\to\infty}\frac{1}{nk}E_{n,k}(X^{(\mathcal{T})}_{n,k}|W_{\mathcal{T}(n,k;R_n)})\ge\frac{1}{\alpha_k}\frac{k}{k+1}-\frac{1-\alpha_k}{\alpha_k}. \tag{4.7}$$

Now (4.5) and (4.7) give (1.7). □

5. Proof of Theorem 2

Proof of Theorem 2. Recall from the proof of Proposition 1 that $A^{(n)}_{M,j,l}$ denotes the event that among the first M items, the number j has occurred l times, and no number greater than j has occurred at all, where $1\le M\le kn-1$, $j\in[n]$ and $l\in[k]$. If $A^{(n)}_{M,j,l}$ occurs, for some $l\in[k]$ and some $j\in[n-1]$, and the strategy $\mathcal{S}(n,k;M)$ succeeds, then by the definition of the strategy, $J^{(\mathcal{S})}_{n,k}=1$. By (2.2), the probability that the strategy $\mathcal{S}(n,k;M)$ succeeds, conditioned on such an $A^{(n)}_{M,j,l}$, is equal to $\frac{k}{k(n-j+1)-l}$.

On the other hand, if $A^{(n)}_{M,n,l}$ occurs, for some $l\in[k-1]$, then by the definition of the strategy, $J^{(\mathcal{S})}_{n,k}=l+1$. Conditioned on $A^{(n)}_{M,n,l}$, it follows by (2.3) that the strategy $\mathcal{S}(n,k;M)$ automatically succeeds. Conditioned on $A^{(n)}_{M,n,k}$, it follows by (2.3) that the strategy $\mathcal{S}(n,k;M)$ automatically fails.

From the considerations in the above two paragraphs, along with (2.1), we have

(5.1)

$$P_{n,k}(J^{(\mathcal{S})}_{n,k}=l|W_{\mathcal{S}(n,k;M_n)})=P_{n,k}(A^{(n)}_{M_n,n,l-1}|W_{\mathcal{S}(n,k;M_n)})=\frac{P_{n,k}(A^{(n)}_{M_n,n,l-1})}{P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}=$$

$$\frac{1}{P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}\frac{\binom{M_n}{l-1}(k)_{l-1}(k(n-1))_{M_n-l+1}}{(kn)_{M_n}},\ l\in\{2,\cdots,k\}.$$

Writing $M_n = c_n kn$ in (5.1), where $\lim_{n\to\infty} c_n = c$, and letting $n \to \infty$ and recalling (3.14), which was derived for $s \ge 1$, but which holds just as well with $s = 0$, it follows that

(5.2)

$$\lim_{n\to\infty} P_{n,k}(J^{(\mathcal{S})}_{n,k} = l|W_{\mathcal{S}(n,k;M_n)}) = \frac{1}{\lim_{n\to\infty} P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}\binom{k}{l-1}(\frac{c}{1-c})^{l-1}(1-c)^k = \frac{1}{\lim_{n\to\infty} P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}\binom{k}{l-1}c^{l-1}(1-c)^{k-l+1},\ l \in \{2, \cdots, k\},$$

which gives (1.4) for $l \in \{2, \cdots, k\}$. Since

$$\sum_{l=2}^{k}\binom{k}{l-1}c^{l-1}(1-c)^{k-l+1} = 1 - c^k - (1-c)^k, \tag{5.3}$$

it follows from (5.2) and (5.3) that (1.4) also holds for $l = 1$. □

6. Proof of Theorem 3

Proof of Theorem 3. Recall that $A^{(n)}_{M,j,l}$ denotes the event that among the first M items, the number j has occurred l times, and no number greater than j has occurred at all, where $1 \le M \le kn - 1$, $j \in [n]$ and $l \in [k]$. We first consider the case that the event $A^{(n)}_{M,j,l}$ occurs, where $j \in [n-1]$, and then afterwards, we consider the case that the event $A^{(n)}_{M,n,l}$ occurs.

Let $j \in [n-1]$. If $A^{(n)}_{M,j,l}$ occurs, then among the last $kn - M$ items, there are $(j-1)k - M + l$ items of rank strictly less than j, there are $k(n-j) - l$ items of ranks between j and $n-1$, and there are k items of rank n.

On the event $W_{\mathcal{S}(n,k;M)} \cap A^{(n)}_{M,j,l}$, the intersection of the event $A^{(n)}_{M,j,l}$ and the event that the strategy $\mathcal{S}(n,k;M)$ successfully selects an item of highest rank n, let $\tau(A^{(n)}_{M,j,l})$ denote the number of additional items after the first M items up until an item of rank n is selected. Then by definition,

$$X^{(\mathcal{S})}_{n,k} = M + \tau(A^{(n)}_{M,j,l}) \text{ on } W_{\mathcal{S}(n,k;M)} \cap A^{(n)}_{M,j,l}. \tag{6.1}$$

From the previous paragraph and from the definition of the strategy $\mathcal{S}(n,k;M)$, the distribution of $\tau(A^{(n)}_{M,j,l})$ under $P_{n,k}(\cdot\ |W_{\mathcal{S}(n,k;M)} \cap A^{(n)}_{M,j,l})$ is the same as the distribution of $\tau^{(N,A)}$ in Proposition 4 with $N = (j-1)k - M + l$ and $A = k$. For convenience, we copy formula (2.2) here:

$$P_{n,k}(W_{\mathcal{S}(n,k;M)}|A^{(n)}_{M,j,l}) = \frac{k}{k(n-j+1)-l},\ j \in [n-1]. \tag{6.2}$$

Now consider the case that the event $A^{(n)}_{M,n,l}$ occurs. Then among the last $kn - M$ items, there are $(n-1)k - M + l$ items of rank strictly less than n, and there are $k - l$ items of rank n.

On the event $W_{\mathcal{S}(n,k;M)} \cap A^{(n)}_{M,n,l}$, the intersection of the event $A^{(n)}_{M,n,l}$ and the event that the strategy $\mathcal{S}(n,k;M)$ successfully selects an item of highest rank n, let $\tau(A^{(n)}_{M,n,l})$ denote the number of additional items after the first M items up until an item of rank n is selected. Then by definition,

$$X^{(\mathcal{S})}_{n,k} = M + \tau(A^{(n)}_{M,n,l}) \text{ on } W_{\mathcal{S}(n,k;M)} \cap A^{(n)}_{M,n,l}. \tag{6.3}$$

From the previous paragraph and from the definition of the strategy $\mathcal{S}(n,k;M)$, the distribution of $\tau(A^{(n)}_{M,n,l})$ under $P_{n,k}(\cdot\ |W_{\mathcal{S}(n,k;M)} \cap A^{(n)}_{M,n,l})$ is the same as the distribution of $\tau^{(N,A)}$ in Proposition 4 with $N = (n-1)k - M + l$ and $A = k - l$. For convenience, we copy formula (2.3) here:

$$P_{n,k}(W_{\mathcal{S}(n,k;M)}|A^{(n)}_{M,n,l}) = \begin{cases} 1, \text{ if } l \in [k-1]; \\ 0, \text{ if } l = k. \end{cases} \tag{6.4}$$

Let $\{M_n\}$ satisfy $M_n = c_n kn$ with $\lim_{n\to\infty} c_n = c \in (0,1)$. Let $H(n,k;M_n)$ denote the expected number of additional items observed after the first M_n items, conditioned on successfully selecting an item of rank n while using strategy $\mathcal{S}(n,k;M_n)$. For ease of notation, let

$$\beta_{n,k,M_n} := P_{n,k}(W_{\mathcal{S}(n,k;M_n)});$$
$$\mathcal{WA}^{(n,k)}_{M_n,j,l} := W_{\mathcal{S}(n,k;M_n)} \cap A^{(n)}_{M_n,j,l}.$$

From the above analysis, it follows that

$$\begin{aligned}
&H(n,k;M_n) = \\
&\frac{1}{\beta_{n,k,M_n}} \sum_{j=1}^{n-1}\sum_{l=1}^{k} P(\mathcal{WA}^{(n,k)}_{M_n,j,l}) E_{n,k}(\tau(A^{(n)}_{M_n,j,l})|\mathcal{WA}^{(n,k)}_{M_n,j,l}) + \\
&\frac{1}{\beta_{n,k,M_n}} \sum_{l=1}^{k} P(\mathcal{WA}^{(n,k)}_{M_n,n,l}) E_{n,k}(\tau(A^{(n)}_{M_n,n,l})|\mathcal{WA}^{(n,k)}_{M_n,n,l}) = \\
&\frac{1}{\beta_{n,k,M_n}} \sum_{j=1}^{n-1}\sum_{l=1}^{k} P(A^{(n)}_{M_n,j,l}) \frac{k}{k(n-j+1)-l} E_{n,k}(\tau(A^{(n)}_{M_n,j,l})|\mathcal{WA}^{(n,k)}_{M_n,j,l}) + \\
&\frac{1}{\beta_{n,k,M_n}} \sum_{l=1}^{k-1} P(A^{(n)}_{M_n,n,l}) E_{n,k}(\tau(A^{(n)}_{M_n,n,l})|\mathcal{WA}^{(n,k)}_{M_n,n,l}),
\end{aligned}$$

where (6.2) and (6.4) were used for the second equality. Substituting from (2.1) in the right hand side of the above equation, and substituting $M_n = c_nkn$, we obtain after division by nk,

(6.5)
$$\frac{1}{nk}H(n,k;M_n)=\frac{1}{\beta_{n,k,c_nkn}}\times$$
$$\sum_{j=1}^{n-1}\sum_{l=1}^{k}\frac{\binom{c_nkn}{l}(k)_l(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{k}{k(n-j+1)-l}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,j,l})|\mathcal{WA}^{(n,k)}_{M_n,j,l})}{nk}+$$
$$\frac{1}{\beta_{n,k,c_nkn}}\sum_{l=1}^{k-1}\frac{\binom{c_nkn}{l}(k)_l(k(n-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,n,l})|\mathcal{WA}^{(n,k)}_{M_n,n,l})}{nk}.$$

Consider first the second sum on the right hand side of (6.5). From earlier in the proof, under $P_{n,k}(\cdot\ |\mathcal{WA}^{(n,k)}_{M_n,n,l})$, the random variable $\tau(A^{(n)}_{c_nkn,n,l})$ has the distribution of $\tau^{N,A}$ from Proposition 4 with $N=(n-1)k-c_nkn+l$ and $A=k-l$. Thus, by Proposition 4,

$$\lim_{n\to\infty}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,n,l})|\mathcal{WA}^{(n,k)}_{M_n,n,l})}{nk}=\frac{1-c}{k-l+1}. \tag{6.6}$$

Using (6.6) and (3.17), it follows that the second term on the right hand side of (6.5) satisfies

(6.7)
$$\lim_{n\to\infty}\frac{1}{\beta_{n,k,c_nkn}}\sum_{l=1}^{k-1}\frac{\binom{c_nkn}{l}(k)_l(k(n-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,n,l})|\mathcal{WA}^{(n,k)}_{M_n,n,l})}{nk}=$$
$$\frac{1}{\lim_{n\to\infty}P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}(1-c)^k\sum_{l=1}^{k-1}\binom{k}{l}(\frac{c}{1-c})^l\frac{1-c}{k-l+1}=$$
$$\frac{1}{\lim_{n\to\infty}P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}(1-c)\sum_{i=2}^{k}\binom{k}{i-1}c^{i-1}(1-c)^{k-i+1}\frac{1}{k+2-i}=$$
$$(1-c)\sum_{i=2}^{k}\frac{p_c^{(k)}(i)}{k+2-i},$$

where the last step follows from (1.4).

Consider now the first sum on the right hand side of (6.5). For $N\in\mathbb{N}$, we break the outer sum $\sum_{j=1}^{n-1}$ into two pieces–the sum from 1 to $n-N$, and the sum from $n-N+1$ to $n-1$. Since $\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,j,l})|\mathcal{WA}^{(n,k)}_{M_n,j,l})}{nk}$ is bounded, it follows from (3.7) that the part of the sum from 1 to $n-N$ satisfies

(6.8)
$$\lim_{N\to\infty}\lim_{n\to\infty}\sum_{j=1}^{n-N}\sum_{l=1}^{k}\frac{\binom{c_nkn}{l}(k)_l(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{k}{k(n-j+1)-l}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,j,l})|\mathcal{WA}^{(n,k)}_{M_n,j,l})}{nk}=0.$$

Letting $s=n-j$, the part of the sum from $n-N+1$ to $n-1$ becomes

(6.9)
$$\sum_{j=n-N+1}^{n-1}\sum_{l=1}^{k}\frac{\binom{c_nkn}{l}(k)_l(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{k}{k(n-j+1)-l}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,j,l})|\mathcal{WA}^{(n,k)}_{M_n,j,l})}{nk}=$$
$$\sum_{s=1}^{N-1}\sum_{l=1}^{k}\frac{\binom{c_nkn}{l}(k)_l(k(n-s-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{k}{k(s+1)-l}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,n-s,l})|\mathcal{WA}^{(n,k)}_{M_n,n-s,l})}{nk}.$$

Recall from earlier in the proof that under $P_{n,k}(\cdot\,|\mathcal{WA}^{(n,k)}_{c_nkn,n-s,l})$, the random variable $\tau(A^{(n)}_{c_nkn,n-s,l})$ has the distribution of $\tau^{N,A}$ from Proposition 4 with $N=(n-s-1)k-c_nkn+l$ and $A=k$. Thus, by Proposition 4,

$$\lim_{n\to\infty}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,n-s,l})|\mathcal{WA}^{(n,k)}_{M_n,n-s,l})}{nk}=\frac{1-c}{k+1}. \tag{6.10}$$

Using (6.9), (6.10) and (3.14) (see also (3.15)), it follows that the first term on the right hand side of (6.5) satisfies

(6.11)
$$\lim_{n\to\infty}\frac{1}{\beta_{n,k,c_nkn}}\times$$
$$\sum_{j=n-N+1}^{n-1}\sum_{l=1}^{k}\frac{\binom{c_nkn}{l}(k)_l(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{k}{k(n-j+1)-l}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,j,l})|\mathcal{WA}^{(n,k)}_{M_n,j,l})}{nk}=$$
$$\frac{1-c}{k+1}\Big(\frac{1}{\lim_{n\to\infty}P_{n,k}(W_{\mathcal{S}(n,k;M_n)})}\sum_{l=1}^{k}\binom{k}{l}(\frac{c}{1-c})^l\sum_{s=1}^{N-1}(1-c)^{k(s+1)}\frac{k}{k(s+1)-l}\Big).$$

A perusal of the current proof, or alternatively, a perusal of the proofs of Proposition 1 and Theorem 1, will reveal that the expression in the large parentheses on the right hand side of (6.11) must be equal to $p_c^{(k)}(1)$. (In the statement and proof of Theorem 2, we calculated $p_c^{(k)}(1)$ more simply as $1-\sum_{i=2}^{k}p_c^{(k)}(i)$.) Substituting this in (6.11), and using (6.8), it follows that

(6.12)

$$\lim_{n\to\infty}\sum_{j=1}^{n-1}\sum_{l=1}^{k}\frac{\binom{c_nkn}{l}(k)_l(k(j-1))_{c_nkn-l}}{(kn)_{c_nkn}}\frac{k}{k(n-j+1)-l}\frac{E_{n,k}(\tau(A^{(n)}_{c_nkn,j,l})|\mathcal{WA}^{(n,k)}_{M_n,j,l})}{nk}=$$

$$(1-c)\frac{p_c^{(k)}(1)}{k+1}.$$

The theorem in the form provided by the first equality in (1.5) follows from (6.1), (6.3), the definition of $H(n,k;M_n)$ in the paragraph following (6.4), (6.5), (6.7) and (6.12). The second equality in (1.5) follows easily from the definition of $\{p_c^{(k)}(i)\}_{i=1}^n$ in (1.4). $\square$

Acknowledgement. The author thanks an anonymous referee for comments that led to the improvement of the presentation of this paper, and for supplying the estimate appearing two paragraphs before the statement of Proposition 1.

References

[1] Bruss, F.T., *Sum the odds to one and stop*, Ann. Probab. **28** (2000), 1384-1391.

[2] Chow, Y.S., Moriguti, S., Robbins, H. and Samuels, S.M, *Optimal selection based on relative rank*, Israel J. Math. **2** (1964), 81-90.

[3] Ferguson, T. *Who solved the secretary problem?*, Stat. Sci. **4** (1989) 282-296.

[4] Freeman, P., *The secretary problem and its extensions: a review*, Internat. Statist. Rev. **51** (1983), 189-206.

[5] Garrod, B., Kubicki, G., and Morayne, M. *How to choose the best twins*, Siam. J. Discrete Math., **26** (2012), 384-398.

[6] Gilbert, J. and Mosteller, F., *Recognizing the maximum of a sequence*, J. Amer. Statist. Assoc. **61** (1966), 35-73.

[7] Hill, T. and Krengel, U., *Minimax-optimal stop rules and distributions in secretary problems*, Ann. Probab. **19** (1991), 342–353.

[8] Hill, T.P. and Kennedy, D.P., *Minimax-optimal strategies for the best-choice problem when a bound is known for the expected number of objects*, SIAM J. Control Optim. **32** (1994), 937–951.

[9] Pfeifer, D., *Extremal processes, secretary problems and the $\frac{1}{e}$ law*, J. Appl. Probab. **26** (1989), 722-733.

[10] Pinsky, R., *The secretary problem with biased arrival order via a Mallows distribution*, Adv. in Appl. Math. **140**, paper no. 102386, (2022), 9 pp.

[11] Pinsky, R., *The secretary problem with non-uniform arrivals via a left-to-right minimum exponentially tilted distribution*, ALEA Lat. Am. J. Probab. Math. Stat. **20** (2023), 1631–1642.

[12] Presman, E.L. and Slonin, I.M. *The best-choice problem for a random number of objects*, Theory Probab. Appl. **17** (1972), 657-668.

Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

Email address: pinsky@math.technion.ac.il

URL: https://pinsky.net.technion.ac.il/