LONGEST SUBSEQUENCE FOR CERTAIN REPEATED UP/DOWN PATTERNS IN RANDOM PERMUTATIONS AVOIDING A PATTERN OF LENGTH THREE

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ABSTRACT. Let S_n denote the set of permutations of $[n]$ and let $\sigma = \sigma_1 \cdots \sigma_n \in S_n$. For any subsequence $\{\sigma_{i_j}\}_{j=1}^k$ of $\{\sigma_i\}_{i=1}^n$ of length $k \ge 2$, construct the "up/down" sequence $V_1 \cdots V_{k-1}$ defined by

$$V_j = \begin{cases} U, \text{ if } \sigma_{i_{j+1}} - \sigma_{i_j} > 0; \\ D, \text{ if } \sigma_{i_{j+1}} - \sigma_{i_j} < 0, \end{cases}$$

where U refers to "up", D to "down" and V to "vertical". Consider now a fixed up/down pattern: $V_1 \cdots V_l$, where $l \in \mathbb{N}$ and $V_j \in \{U, D\}$, $j \in [l]$. Given a permutation $\sigma \in S_n$, consider the length of the longest subsequence of σ that repeats this pattern. Incomplete patterns are not counted, so the length is necessarily either 0 or of the form $kl+1$, where $k \in \mathbb{N}$. For example, consider $l = 3$ and $V_1V_2V_3 = UUD$. Then for the permutation $342617985 \in S_9$, the length of the longest subsequence that repeats the pattern UUD is 7; it is obtained by three different subsequences, namely 3461798, 3461795 and 3461785.

The above framework includes one much studied case as well as another case that has been studied to some degree. The pattern U is the celebrated case of the longest increasing subsequence. The pattern UD (or DU) is the case of the longest alternating subsequence. These have been studied both under the uniform distribution on S_n as well as under the uniform distribution on those permutations in S_n which avoid a particular pattern of length three.

In this paper, we consider the patterns UUD and $UUUD$ under the uniform distribution on those permutations in S_n that avoid the pattern 132. We prove that the expected value of the longest increasing subsequence following the pattern UUD is asymptotic to $\frac{3}{7}n$ and the expected value of the longest increasing subsequence following the pattern $UUUD$ is asymptotic to $\frac{4}{11}n$. (For UD (alternating subsequences) it is known to be $\frac{1}{2}n$.) This leads directly to appropriate corresponding results for permutations avoiding any particular pattern of length three.

2010 *Mathematics Subject Classification.* 60C05, 05A05.

1. Introduction and Statement of Results

Let S_n denote the set of permutations of $[n] = \{1, \cdots, n\}$ and let $\sigma = \sigma_1 \cdots \sigma_n \in S_n$. For any subsequence $\{\sigma_{i_j}\}_{j=1}^k$ of $\{\sigma_i\}_{i=1}^n$ of length $k \geq 2$, construct the "up/down" sequence $V_1 \cdots V_{k-1}$ defined by

$$V_j = \begin{cases} U, \text{ if } \sigma_{i_{j+1}} - \sigma_{i_j} > 0; \\ D, \text{ if } \sigma_{i_{j+1}} - \sigma_{i_j} < 0, \end{cases}$$

where U refers to "up", D refers to "down" and V refers to "vertical".

Consider now a fixed up/down pattern: $V_1 \cdots V_l$, where $l \in \mathbb{N}$ and $V_j \in \{U, D\},\ j \in [l]$. Given a permutation $\sigma \in S_n$, consider the length of the longest subsequence of σ that repeats this pattern. Incomplete patterns are not counted, so the length is necessarily either 0 or of the form $kl + 1$, where $k \in \mathbb{N}$. For example, consider $l = 3$ and $V_1V_2V_3 = UUD$. Then for the permutation $342617985 \in S_9$, the length of the longest subsequence that repeats the pattern UUD is 7; it is obtained by three different subsequences, namely 3461798, 3461795 and 3461785. On the other hand, for the permutation 319652478, the length of the longest subsequence that repeats the pattern UUD is 0 because this pattern does not appear at all.

The above framework includes one very celebrated and much studied case as well as another case that has been studied to some degree. The pattern U is the case of the longest increasing subsequence. This celebrated case was studied by Logan and Shepp [8] and Vershik and Kerov [17]. Their work showed that the expected value of the length of the longest increasing subsequence in a uniformly random permutation from S_n behaves asymptotically as $2\sqrt{n}$. More precise information on the behavior of this random variable was obtained later in the seminal paper of Baik, Deift and Johansson [2]; for more on the longest increasing subsequence and many additional references, see the book by Romik [14].

The pattern UD (or DU) is the case of the longest alternating subsequence (in the first case starting with increasing and ending with decreasing, and in the second case vice versa). Stanley [16] investigated alternating sequences and showed that the expected value of the longest alternating subsequence in a uniformly random permutation from S_n behaves asymptotically as $\frac{2}{3}n$. (Of course this asymptotic behavior is independent of how we define the initial or terminal direction in the sequence.) See also further results by Widom [18].

The analysis in the alternating case is simpler than in the increasing case because, as Stanley noted, there is always a longest alternating subsequence (either beginning with down or ending with up) of $\sigma \in S_n$ which contains the number n. Thus, a longest subsequence can be broken down into smaller pieces that are concatenated. The methods of Stanley and of Widom are very much combinatorial. For a probabilistic approach to the longest alternating sequence, see [7] and [13].

In [16], Stanley also posed the question of whether it is true that for any pattern of ups and downs as we have defined above, there exist constants μ and c such that the expected value of the length of the longest subsequence repeating this pattern in a uniformly random permutation from S_n behaves asymptotically as μn^c. The recent paper [1] answered this question in the affirmative, and showed in particular that for every pattern except for the pattern U corresponding to the longest increasing subsequence, one has $c = 1$. Thus, for every pattern except for U, the asymptotic behavior of the expected value of the length of the longest subsequence repeating that pattern is μn, for some $\mu \in (0, 1)$. The authors of [1] did not explicitly calculate the value of μ; however, they constructed a dynamical system that can be used to approximate μ. They also proved a central limit theorem for the length of the longest subsequence repeating any particular pattern, except for the pattern U.

In this paper, instead of considering a permutation that is uniformly random from S_n, we consider a permutation that is uniformly random from those permutations in S_n which avoid a certain pattern of length three. Our aim is to calculate explicitly the asymptotic behavior of the expected value of the length of the longest subsequence repeating certain patterns of ups and downs in such a random permutation. Before continuing to explain this, we recall the definition of pattern-avoiding for permutations. If $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ and $\tau = \tau_1 \cdots \tau_m \in S_m$, where $2 \leq m < n$, then we say that σ contains τ as a pattern if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that for all $1 \leq j, k \leq m$, the inequality $\sigma_{i_j} < \sigma_{i_k}$ holds if and only if the inequality $\tau_j < \tau_k$ holds. If σ does not contain τ, then we say that σ avoids τ. We consider here permutations on S_n that avoid a pattern $\tau \in S_3$. Denote by $S_n(\tau)$ the set of permutations in S_n that avoid τ. It is well-known that $|S_n(\tau)| = C_n$, for all six permutations $\tau \in S_3$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$, $n \in \mathbb{N}$, is the nth Catalan number [3]. Let $P_n^{\text{av}(\tau)}$ denote the uniform probability measure on $S_n(\tau)$ and let $E_n^{\text{av}(\tau)}$ denote the corresponding expectation.

As already noted, the pattern U corresponds to increasing subsequences. In [4], the asymptotic behavior of the expectation of the longest increasing subsequence L_n of a random permutation under the distribution $P_n^{\text{av}(\tau)}$ was obtained for all six permutations $\tau \in S_3$. Of course, the case $\tau = 123$ is trivial. The expectation is on the order n only for $\tau \in \{231, 312, 321\}$. The asymptotic behavior of the variance $v_n(\tau)$ was also investigated, and the limiting distribution of $\frac{L_n - E_n^{\text{av}(\tau)} L_n}{v_n(\tau)}$ was calculated, the limit being Gaussian only for $\tau \in \{231, 312\}$. Large deviations were considered in [11]. For other results concerning longest increasing subsequences in pattern-avoiding permutations, see [9] and [10].

As already noted, the pattern UD (or DU) corresponds to alternating subsequences. In [6], the asymptotic behavior of the expectation of the longest alternating sequence of a random permutation under the distribution $P_n^{\text{av}(\tau)}$ was shown to be $\frac{n}{2}$ for all six choices of $\tau \in S_n$. The asymptotic variance was also obtained as well as a central limit theorem. Large deviations were considered in [11].

In this paper, for the patterns UUD and $UUUD$, we will calculate the asymptotic behavior of the expectation of the longest subsequence repeating that pattern in a uniformly random permutation avoiding the pattern 132. The proof in the case of UUD involves analyzing the asymptotic behavior of the coefficients of either of two generating functions that satisfy a system of two linear equations. The calculations are somewhat involved. The proof in the case $UUUD$ involves analyzing the asymptotic behavior of the coefficients of any one of three generating functions that satisfy a system of three linear equations. Here the calculations are quite involved. In principle, our technique can be continued for the pattern $U^l D$, for any $l \in \mathbb{N}$, where U^l indicates l consecutive U's. However, this involves solving a system of l linear equations for l generating functions, solving explicitly for one of them, and then analyzing its coefficients. It also involves solving an auxiliary set of equations to calculate the probability that $\sigma \in S_n(132)$ does not have an increasing subsequence of length j, for $j = 1, \cdots, l$ (an extension of Lemma 1 in Section 3).

We now state two theorems for 132-avoiding permutations, one for the pattern UUD and one for the pattern $UUUD$.

Theorem 1. *Let $L_n^{U^2D}(\sigma)$ denote the length of the longest subsequence of the repeated pattern UUD in $\sigma \in S_n(132)$. (So $L_n^{U^2D}(\sigma)$ is either equal to 0 or to $3k+1$ for some $k \in \mathbb{N}$.) Then*

$$E_n^{av(132)} L_n^{U^2D} \sim \frac{3}{7}n. \tag{1.1}$$

Theorem 2. *Let $L_n^{U^3D}(\sigma)$ denote the length of the longest subsequence of the repeated pattern $UUUD$ in $\sigma \in S_n(132)$. (So $L_n^{U^3D}(\sigma)$ is either equal to 0 or to $4k+1$ for some $k \in \mathbb{N}$.) Then*

$$E_n^{av(132)} L_n^{U^3D} \sim \frac{4}{11}n. \tag{1.2}$$

Remark. Recall that for the repeated pattern UD (which corresponds to alternating subsequences), the corresponding asymptotic behavior is $\frac{1}{2}n$.

The reason the cases U^lD are in principle tractable for 132-avoiding permutations is that a variant of Stanley's observation holds in these cases; namely, that for a 132-avoiding permutation, there is always either a longest subsequence repeating the pattern U^lD that contains the number n, or else, every such longest subsequence starts after the appearance of the number n in the permutation. Using the reversal, complementation, and reversal-complementation operations for permutations, the results we obtain for the up-down patterns UUD and $UUUD$ for permutations avoiding the pattern 132 can be translated into similar results for appropriate up-down patterns for permutations avoiding any one of the patterns $213, 231, 312$. Using a well-known bijection between permutations avoiding the pattern 132 and permutations avoiding the pattern 123, along with reversal, the results we obtain can also be translated into similar results for appropriate up-down patterns for permutations avoiding either of the patterns 123, 321. These results are given in the following corollary.

Corollary 1. *i. Let $L_n^{V_1V_2V_3}(\sigma)$ denote the length of the longest subsequence of the repeated pattern $V_1V_2V_3$ in $\sigma \in S_n$, where $V_i \in \{U, D\}$, $i = 1, 2, 3$.*

Then

$$E_n^{av(\tau)} L_n^{V_1V_2V_3} \sim \frac{3}{7}n, \tag{1.3}$$

for the following five pairs of $V_1V_2V_3$ and τ: UDD and 231; DDU and 312; DUU and 213; UDD and 123; UUD and 321.
ii. Let $L_n^{V_1V_2V_3V_4}(\sigma)$ denote the length of the longest subsequence of the repeated pattern $V_1V_2V_3V_4$ in $\sigma \in S_n$, where $V_i \in \{U, D\}$, $i = 1, 2, 3, 4$. Then

$$E_n^{av(\tau)} L_n^{V_1V_2V_3V_4} \sim \frac{4}{11}n, \tag{1.4}$$

for the following five pairs of $V_1V_2V_3$ and τ: $UDDD$ and 231; $DDDU$ and 312; $DUUU$ and 213; $UDDD$ and 123; $UUUD$ and 321.

We prove Corollary 1 in section 5.

We prove Theorem 1 in Section 2. We derive a system of two linear equations for two generating functions, and then solve explicitly for one of them. These generating functions are connected to the expected number of complete patterns UUD in a maximal subsequence. The leading order asymptotic behavior of the coefficients of either of these generating functions is equal to the leading order asymptotic behavior of $\frac{1}{3}C_n E_n^{\text{av}(132)} L_n^{U^2D}$. Performing an asymptotic analysis of the coefficients of this generating function yields the proof of the theorem.

The proof of Theorem 2 is much longer. In Section 3 we derive a system of three linear equations for three generating functions, and then solve explicitly for one of them. The explicit expression for this generating function is quite involved. These generating functions are connected to the expected number of complete patterns $UUUD$ in a maximal subsequence. The leading order asymptotic behavior of the coefficients of any of these three generating functions is equal to the leading order asymptotic behavior of $\frac{1}{4}C_n E_n^{\text{av}(132)} L_n^{U^3D}$. In Section 4 we perform a lot of algebra in order to obtain the generating function in a more manageable form. Then we perform an asymptotic analysis of the coefficients of this generating function to yield the proof of the theorem.

2. Proof of Theorem 1

For $\sigma \in S_n$ and $n \in \mathbb{N}$, define $B_n^{U^2D}(\sigma)$ to be the number of complete sets of UUD in a longest subsequence in σ of the repeated pattern UUD. Thus,

$$B_n^{U^2D}(\sigma) = \begin{cases} \frac{1}{3}\left(L_n^{U^2D}(\sigma) - 1\right), \text{ if } L_n^{U^2D}(\sigma) \neq 0; \\ 0, \text{if } L_n^{U^2D}(\sigma) = 0. \end{cases} \tag{2.1}$$

Also, for convenience, we define $B_0^{U^2D} \equiv 0$.

For $\sigma \in S_n$ and $n \in \mathbb{N}$, define $A_n^{U^2D}(\sigma) = 0$, if $\sigma = n \cdots 21$; otherwise, find a longest subsequence $\{\sigma_{i_j}\}_{j=1}^{3k+2}$, $k \in \mathbb{Z}^+$, of σ for which the up/down pattern is $UUD \cdots UUDU$, and define $A_n^{U^2D}(\sigma) = k+1$. For convenience, we define $A_0^{U^2D} \equiv 0$.

In the sequel, for any $j \in \mathbb{N}$, $B_j^{U^2D}$ and $A_j^{U^2D}$ will always be considered as random variables on the probability space $\left(S_j(132), P_j^{\text{av}(132)}\right)$. Define

$$b_n = E_n^{\text{av}(132)} B_n^{U^2D}; \quad a_n = E_n^{\text{av}(132)} A_n^{U^2D}, \tag{2.2}$$

where we have suppressed the notation U^2D. Define the generating functions for $\{C_n b_n\}_{n=0}^{\infty}$ and $\{C_n a_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} \mathcal{B}^{U^2D}(t) &= \sum_{n=0}^{\infty} C_n b_n t^n; \\ \mathcal{A}^{U^2D}(t) &= \sum_{n=0}^{\infty} C_n a_n t^n. \end{aligned} \tag{2.3}$$

Also let $C(t) = \sum_{n=0}^{\infty} C_n t^t$ denote the generating function of the Catalan numbers, where we define $C_0 = 1$. As is well-known,

$$C(t) = \frac{1 - \sqrt{1-4t}}{2t}. \tag{2.4}$$

The following definition will be useful. Let $a_1 < a_2 \cdots < a_m$ be real numbers and let $\rho = \rho_1 \cdots \rho_m$ be a permutation of these numbers. We define $\text{red}(\rho) \in S_m$, the reduction of ρ, to be the permutation in S_m that has the same pattern as ρ. That is, $\text{red}(\rho) = \sigma$ if σ satisfies $\sigma_i < \sigma_j$ whenever $\rho_i < \rho_j$, $i, j \in [m]$. Note that the up/down pattern that one can associate with $\rho = \rho_1 \cdots \rho_m$ is the same as the up/down pattern associated with $\text{red}(\rho)$. Every permutation $\sigma \in S_n(132)$ has the property that if $\sigma_j = n$, then the numbers $\{n-j+1, \cdots, n-1\}$ appear in the first $j-1$ positions in σ and the numbers $\{1,\cdots,n-j\}$ appear in the last $n-j$ positions in σ. From this fact, along with the fact that $|S_n(132)|=C_n$, it follows that

$$P_n^{\text{av}(132)}(\sigma_j=n)=\frac{C_{j-1}C_{n-j}}{C_n},\ \text{for } j\in[n]. \tag{2.5}$$

It also follows that under the conditioned measure $P_n^{\text{av}(132)}|\{\sigma_j=n\}$, the permutation $\text{red}(\sigma_1\cdots\sigma_{j-1})\in S_{j-1}$ has the distribution $P_{j-1}^{\text{av}(132)}$, the permutation $\sigma_{j+1}\cdots\sigma_n\in S_{n-j}$ has the distribution $P_{n-j}^{\text{av}(132)}$, and these two permutations are independent.

We now derive a system of two linear equations for $\mathcal{B}^{U^2D}(t)$ and $\mathcal{A}^{U^2D}(t)$, and then solve for one of them explicitly. From the definitions of $B_n^{U^2D}$ and $A_n^{U^2D}$, we have

$$B_n^{U^2D}\equiv 0,\ 0\le n\le 3;\quad A_n^{U^2D}\equiv 0,\ 0\le n\le 1.$$

Thus,

$$\begin{aligned} &b_n=0,\ 0\le n\le 3;\\ &a_n=0,\ 0\le n\le 1. \end{aligned} \tag{2.6}$$

The following proposition is the key to obtaining a pair of linear equations for the generating functions $\mathcal{B}^{U^2D}(t)$ and $\mathcal{A}^{U^2D}(t)$.

Proposition 1. *i.*

$$\begin{aligned} &B_n^{U^2D}|\{\sigma_j=n\}\stackrel{dist}{=}A_{j-1}^{U^2D}+B_{n-j}^{U^2D},\ j\in[n-1],\ n\ge 2;\\ &B_n^{U^2D}|\{\sigma_n=n\}\stackrel{dist}{=}B_{n-1}^{U^2D},\ n\ge 2, \end{aligned} \tag{2.7}$$

where on the right hand side of (2.7), $A_{j-1}^{U^2D}$ *and* $B_{n-j}^{U^2D}$ *are understood to be independent.*

ii.

(2.8)

$$\begin{aligned} &A_n^{U^2D}|\{\sigma_1=n\}\stackrel{dist}{=}A_{n-1}^{U^2D},\ n\ge 2;\\ &A_n^{U^2D}|\{\sigma_j=n\}\stackrel{dist}{=}\left(A_{j-1}^{U^2D}+A_{n-j}^{U^2D}\right)1_{\{A_{n-j}^{U^2D}\neq 0\}}+\left(B_{j-1}^{U^2D}+1\right)1_{\{A_{n-j}^{U^2D}=0\}},\\ &j\in\{2,\cdots,n\},\ n\ge 2, \end{aligned}$$

where on the right hand side of (2.8), $A_{j-1}^{U^2D}$ *and* $A_{n-j}^{U^2D}$ *are understood to be independent and* $B_{j-1}^{U^2D}$ *and* $A_{n-j}^{U^2D}$ *are understood to be independent.*

Proof. The first line of (2.7) follows from the equality

$$\begin{aligned} &B_n^{U^2D}(\sigma) = A_{j-1}^{U^2D}(\text{red}(\sigma_1 \cdots \sigma_{j-1})) + B_{n-j}^{U^2D}(\sigma_{j+1} \cdots \sigma_n), \text{ if } \sigma_j = n,\\ &\text{for } j \in [n-1], n \geq 2, \end{aligned} \tag{2.9}$$

along with the fact noted above that under the conditioned measure $P_n^{\text{av}(132)}|\{\sigma_j = n\}$, the permutation $\text{red}(\sigma_1 \cdots \sigma_{j-1}) \in S_{j-1}$ has the distribution $P_{j-1}^{\text{av}(132)}$, the permutation $\sigma_{j+1} \cdots \sigma_n \in S_{n-j}$ has the distribution $P_{n-j}^{\text{av}(132)}$, and these two permutations are independent. Rather than give a formal proof of (2.9), we convince the reader of its validity by giving an example and then a generic explanation.

Let $\sigma = 435768921$. Then $n = 9$ and $j = 7$. We have $A_{j-1}^{U^2D}(\text{red}(\sigma_1 \cdots \sigma_{j-1})) = A_6^{U^2D}(\text{red}(435768)) = A^{U^2D}(213546) = 2$, because the subsequence 23546 (as well as 13546) corresponds to $UUDU$. We have $B_{n-j}^{U^2D}(\sigma_{j+1} \cdots \sigma_n) = B_2^{U^2D}(21) = 0$. And we have $B_n^{U^2D}(\sigma) = B_9^{U^2D}(435768921) = 2$ because the subsequence 4576892 (as well as several others) corresponds to $UUDUUD$.

Generically, $B_n^{U^2D}(\sigma)$ is the sum of two terms. One of the terms is $A_{j-1}^{U^2D}(\text{red}(\sigma_1 \cdots \sigma_{j-1}))$, which counts the number of full sets of UUD and then adds one for an extra U. This extra U, along with $\sigma_j = n$ and σ_{j+1} supply an additional full set UUD which was counted by $A_{j-1}^{U^2D}(\text{red}(\sigma_1 \cdots \sigma_{j-1}))$ (via the adding one for the extra U). The other term is $B_{n-j}^{U^2D}(\sigma_{j+1} \cdots \sigma_n)$, which counts the remaining sets of UUD.

The second line of (2.7) is obtained using the following rather obvious equality instead of (2.9):

$$B_n^{U^2D}(\sigma) = B_{n-1}^{U^2D}(\sigma_1 \cdots \sigma_{n-1}), \text{ if } \sigma_n = n.$$

The second line of (2.8) follows from the equality

(2.10)

$$A_n^{U^2D}(\sigma) = \Big(A_{j-1}^{U^2D}(\text{red}(\sigma_1 \cdots \sigma_{j-1})) + A_{n-j}^{U^2D}(\sigma_{j+1} \cdots \sigma_n)\Big) 1_{A_{n-j}^{U^2D}(\sigma_{j+1}\cdots\sigma_n)\neq 0} +$$
$$\Big(B_{j-1}^{U^2D}(\text{red}(\sigma_1 \cdots \sigma_{j-1})) + 1\Big) 1_{A_{n-j}^{U^2D}(\sigma_{j+1}\cdots\sigma_n)=0}, \text{ if } \sigma_j = n, \text{ for } j \in [n-1], n \geq 2,$$

along with the fact noted above that under the conditioned measure $P_n^{\text{av}(132)}|\{\sigma_j = n\}$, the permutation $\text{red}(\sigma_1 \cdots \sigma_{j-1}) \in S_{j-1}$ has the distribution $P_{j-1}^{\text{av}(132)}$, the permutation $\sigma_{j+1} \cdots \sigma_n \in S_{n-j}$ has the distribution $P_{n-j}^{\text{av}(132)}$, and these two permutations are independent.

In the case that $A^{U^2D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)\neq 0$, (2.10) is obtained by reasoning similar to that for (2.9). We explain (2.10) in the case that $A^{U^2D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=0$ with an example. Let $\sigma=435786921$ (slightly different than the σ used above). So $n=9$ and $j=7$. We have $A^{U^2D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=A^{U^2D}_{2}(21)=0$. We have $B^{U^2D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))=B^{U^2D}_{6}(\text{red}(435786))=B^{U^2D}_{6}(213564)=1$ because the subsequence 2354 (as well as several others) corresponds to UUD. And we have $A^{U^2D}_{n}(\sigma)=A^{U^2D}_{9}(435786921)=2$ because the subsequence 45769 (as well as several others) corresponds to $UUDU$. (Note that $A^{U^2D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))=A^{U^2D}_{6}(\text{red}(435786))=A^{U^2D}_{6}(213564)=1$ because the subsequence 23 (as well as several others) corresponds to U. Thus, when $A^{U^2D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=0$, it is not true in general that $A^{U^2D}_{n}(\sigma)=A^{U^2D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))+A^{U^2D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)$.)

The first line of (2.8) is obtained using the following rather obvious equality instead of (2.10):

$$A^{U^2D}_{n}(\sigma)=A^{U^2D}_{n-1}(\sigma_2\cdots\sigma_n),\ \text{ if } \sigma_1=n.$$

□

From (2.5) and (2.7), it follows that

(2.11)

$$b_n=E^{\text{av}(132)}_n B^{U^2D}_n=\sum_{j=1}^{n}E^{\text{av}(132)}_n(B^{U^2D}_n|\sigma_j=n)P^{\text{av}(132)}_n(\sigma_j=n)=$$
$$\sum_{j=1}^{n-1}\Big(E^{\text{av}(132)}_{j-1}A^{U^2D}_{j-1}+E^{\text{av}(132)}_{n-j}B^{U^2D}_{n-j}\Big)\frac{C_{j-1}C_{n-j}}{C_n}+E^{\text{av}(132)}_{n-1}B^{U^2D}_{n-1}\frac{C_{n-1}C_0}{C_n}=$$
$$\sum_{j=1}^{n-1}(a_{j-1}+b_{n-j})\frac{C_{j-1}C_{n-j}}{C_n}+b_{n-1}\frac{C_{n-1}C_0}{C_n},\ n\geq 2.$$

Multiplying both sides of (2.11) by C_nt^n, summing over n from 4 to ∞, and using (2.6), we obtain

(2.12)

$$\mathcal{B}^{U^2D}(t)=\sum_{n=4}^{\infty}C_nb_nt^n=t\sum_{n=4}^{\infty}\left(\sum_{j=1}^{n-1}(a_{j-1}+b_{n-j})\,C_{j-1}C_{n-j}\right)t^{n-1}+$$
$$t\sum_{n=4}^{\infty}b_{n-1}C_{n-1}t^{n-1}.$$

Straightforward algebraic calculations along with (2.6) show that

$$\begin{aligned}
&\sum_{n=4}^{\infty}\left(\sum_{j=1}^{n-1} a_{j-1}C_{j-1}C_{n-j}\right)t^{n-1} = \mathcal{A}^{U^2D}(t)\left(C(t)-1\right); \\
&\sum_{n=4}^{\infty}\left(\sum_{j=1}^{n-1} b_{n-j}C_{j-1}C_{n-j}\right)t^{n-1} = \mathcal{B}^{U^2D}(t)C(t).
\end{aligned} \tag{2.13}$$

From (2.12) and (2.13), we obtain

$$\mathcal{B}^{U^2D}(t) = t\left(\mathcal{A}^{U^2D}(t)\left(C(t)-1\right) + \mathcal{B}^{U^2D}(t)C(t) + \mathcal{B}^{U^2D}(t)\right),$$

which we write as

$$\mathcal{B}^{U^2D}(t) = \frac{t\left(C(t)-1\right)\mathcal{A}^{U^2D}(t)}{1-t-tC(t)}. \tag{2.14}$$

Note that for $l \in \mathbb{N}$ and $\sigma \in S_l(132)$, $A_l^{U^2D}(\sigma) = 0$ only for $\sigma = l\cdots 21$; thus $P_l^{\text{av}(132)}(A_l^{U^2D} = 0) = \frac{1}{C_l}$. Using this with (2.5) and (2.8), it follows that

$$\begin{aligned}
&a_n = E_n^{\text{av}(132)}A_n^{U^2D} = \sum_{j=1}^{n} E_n^{\text{av}(132)}(\mathcal{A}_n^{U^2D}|\sigma_j = n)P_n^{\text{av}(132)}(\sigma_j = n) = \\
&\frac{C_0C_{n-1}}{C_n}a_{n-1} + \sum_{j=2}^{n}\left(a_{j-1}\left(1-\frac{1}{C_{n-j}}\right) + a_{n-j}\right)\frac{C_{j-1}C_{n-j}}{C_n} + \\
&\sum_{j=2}^{n}\frac{b_{j-1}+1}{C_{n-j}}\,\frac{C_{j-1}C_{n-j}}{C_n}.
\end{aligned} \tag{2.15}$$

Multiplying both sides of (2.15) by C_nt^n, summing over n from 2 to ∞ and using (2.6), we obtain

$$\begin{aligned}
&\mathcal{A}^{U^2D}(t) = \sum_{n=2}^{\infty} C_na_nt^n = \\
&t\sum_{n=2}^{\infty} C_{n-1}a_{n-1}t^{n-1} + t\sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} a_{j-1}C_{j-1}C_{n-j}\right)t^{n-1} - t\sum_{n=2}^{\infty}\big(\sum_{j=2}^{n} a_{j-1}C_{j-1}\big)t^{n-1} + \\
&t\sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} C_{j-1}a_{n-j}C_{n-j}\right)t^{n-1} + t\sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} b_{j-1}C_{j-1}\right)t^{n-1} + t\sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} C_{j-1}\right)t^{n-1}.
\end{aligned} \tag{2.16}$$

Straightforward algebraic calculations along with (2.6) show that

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} a_{j-1}C_{j-1}C_{n-j}\right)t^{n-1} = \mathcal{A}^{U^2D}(t)C(t);\\
&\sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} C_{j-1}a_{n-j}C_{n-j}\right)t^{n-1} = \mathcal{A}^{U^2D}(t)\left(C(t)-1\right);\\
&\sum_{n=2}^{\infty}\big(\sum_{j=2}^{n} a_{j-1}C_{j-1}\big)t^{n-1} = \frac{\mathcal{A}^{U^2D}(t)}{1-t};\\
&\sum_{n=2}^{\infty}\big(\sum_{j=2}^{n} b_{j-1}C_{j-1}\big)t^{n-1} = \frac{\mathcal{B}^{U^2D}(t)}{1-t};\\
&\sum_{n=2}^{\infty}\big(\sum_{j=2}^{n} C_{j-1}\big)t^{n-1} = \frac{C(t)-1}{1-t}.
\end{aligned}
\tag{2.17}
$$

From (2.16) and (2.17), we obtain

$$
\mathcal{A}^{U^2D}(t) = t\Bigg(\mathcal{A}^{U^2D}(t)+\mathcal{A}^{U^2D}(t)C(t)-\frac{\mathcal{A}^{U^2D}(t)}{1-t}+\mathcal{A}^{U^2D}(t)\left(C(t)-1\right)+
\frac{\mathcal{B}^{U^2D}(t)}{1-t}+\frac{C(t)-1}{1-t}\Bigg),
\tag{2.18}
$$

which we write as

$$
\mathcal{A}^{U^2D}(t) = \frac{t\left(\mathcal{B}^{U^2D}(t)+C(t)-1\right)}{(1-t)\left(1-2tC(t)\right)+t}.
\tag{2.19}
$$

Substituting (2.19) in (2.14) and solving for $\mathcal{B}^{U^2D}(t)$, we obtain

$$
\mathcal{B}^{U^2D}(t) = \frac{t^2\left(C(t)-1\right)^2}{\left((1-t)(1-2tC(t))+t\right)\left(1-t-tC(t)\right)-t^2(C(t)-1)}.
\tag{2.20}
$$

We write the denominator in (2.20) as

$$
\begin{aligned}
&\left((1-t)(1-2tC(t))+t\right)\left(1-t-tC(t)\right)-t^2(C(t)-1) =\\
&t^2-t+1+(-2t^3+3t^2-3t)C(t)+2t^2(1-t)C^2(t).
\end{aligned}
\tag{2.21}
$$

Using (2.4) and performing some algebra, we have

$$
\begin{aligned}
&t^2-t+1+(-2t^3+3t^2-3t)C(t)+2t^2(1-t)C^2(t) =\\
&\frac{1}{2}\left((2t^2-t+1)\sqrt{1-4t}+(1-t)(1-4t)\right).
\end{aligned}
\tag{2.22}
$$

Using (2.4), the numerator in (2.20) can be written as.

$$t^2(C(t)-1)^2 = \frac{1}{2}\left(2t^2-4t+1+(2t-1)\sqrt{1-4t}\right). \tag{2.23}$$

From (2.20)-(2.23), we obtain

$$\mathcal{B}^{U^2D}(t) = \frac{2t^2-4t+1+(2t-1)\sqrt{1-4t}}{(2t^2-t+1)\sqrt{1-4t}+(1-t)(1-4t)}. \tag{2.24}$$

The denominator on the right hand side above is defined for $t \le \frac{1}{4}$ and vanishes only at $t=\frac{1}{4}$. Let $x=(1-4t)^{\frac{1}{2}} \ge 0$. Then $t=\frac{1-x^2}{4}$. We can rewrite $\mathcal{B}^{U^2D}(t)$ in terms of x as

$$\mathcal{B}^{U^2D}(t) = \frac{1}{7x}\left(\frac{7-28x+42x^2-28x^3+7x^4}{7+6x+2x^3+x^4}\right),\ x=\sqrt{1-4t}. \tag{2.25}$$

It is well known (see for example, [5, p. 381]) that

$$[t^n]\,(1-4t)^{-\alpha} = 4^n\frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+O\left(\frac{1}{n}\right)\right),\ \text{for } \alpha\in\mathbb{C}-\mathbb{Z}_{\le 0}. \tag{2.26}$$

where as usual, $[t^n]f(t)$ denotes the coefficient of t^n in the power series representation of a function $f(t)$. Denote the expression multiplying $\frac{1}{7x}$ on the right hand side of (2.25) by $g(x)$, and now consider x to be a complex variable. Then $g(x)$ is analytic in a neighborhood of zero, so one has

$$g(x) = \frac{7-28x+42x^2-28x^3+7x^4}{7+6x+2x^3+x^4} = \sum_{m=0}^{\infty} a_m x^m,\ a_0=1, \tag{2.27}$$

and there exists an R such that $|a_m| \le R^m$. From (2.25) and (2.27), we write

$$\mathcal{B}^{U^2D}(t) = \frac{1}{7x} + \sum_{m=0}^{\infty}\frac{1}{7}a_{m+1}x^m$$

From (2.25), (2.26) and the definition of x, it follows that

$$[t^n]\mathcal{B}^{U^2D}(t) \sim \frac{1}{7}4^n\frac{n^{-\frac{1}{2}}}{\sqrt{\pi}}. \tag{2.28}$$

From (2.3) and (2.2), we have $[t^n]\mathcal{B}^{U^2D}(t) = C_n b_n = C_n E_n^{\mathrm{av}(132)} B_n^{U^2D}$, and as is well known, the Catalan numbers satisfy $C_n \sim 4^n\frac{n^{-\frac{3}{2}}}{\sqrt{\pi}}$. Using these facts with (2.28), we conclude that

$$E_n^{\mathrm{av}(132)} B_n^{U^2D} \sim \frac{1}{7}n. \tag{2.29}$$

Now Theorem 1 follows from (2.29) and (2.1). □

3. Derivation of the generating functions for Theorem 2

For $\sigma \in S_n$ and $n \in \mathbb{N}$, define $B_n^{U^3D}(\sigma)$ to be the number of complete sets of $UUUD$ in a longest subsequence in σ of the repeated pattern $UUUD$. Thus,

$$B_n^{U^3D}(\sigma) = \begin{cases} \frac{1}{4}\left(L_n^{U^3D}(\sigma) - 1\right), \text{ if } L_n^{U^3D}(\sigma) \neq 0; \\ 0, \text{if } L_n^{U^2D}(\sigma) = 0. \end{cases} \tag{3.1}$$

Also, for convenience, we define $B_0^{U^3D} \equiv 0$.

For $\sigma \in S_n$ and $n \in \mathbb{N}$, define $G_n^{U^3D}(\sigma) = 0$, if $\sigma = n \cdots 21$; otherwise, find a longest subsequence $\{\sigma_{i_j}\}_{j=1}^{4k+2}$, $k \in \mathbb{Z}^+$, of σ for which the up/down pattern is $UUUD \cdots UUUDU$, and define $G_n^{U^3D}(\sigma) = k+1$. For convenience, we define $G_0^{U^3D} \equiv 0$.

For $\sigma \in S_n$ and $n \in \mathbb{N}$, define $A_n^{U^3D}(\sigma) = 0$, if σ has no increasing subsequence of length three (or equivalently, if σ has no subsequence $\{\sigma_{i_j}\}_{j=1}^{3}$ which corresponds to the pattern UU); otherwise, find a longest subsequence $\{\sigma_{i_j}\}_{j=1}^{4k+3}$, $k \in \mathbb{Z}^+$, of σ for which the up/down pattern is $UUUD \cdots UUUDUU$, and define $A_n^{U^3D}(\sigma) = k+1$. For convenience, we define $A_0^{U^3D} \equiv 0$.

In the sequel, for $j \in \mathbb{N}$, $B_j^{U^3D}, G_j^{U^3D}$ and $A_j^{U^3D}$ will always be considered as random variables on the probability space $\left(S_j(132), P_j^{\text{av}(132)}\right)$. Define

$$b_n = E_n^{\text{av}(132)} B_n^{U^3D};\ g_n = E_n^{\text{av}(132)} G_n^{U^3D};\ a_n = E_n^{\text{av}(132)} A_n^{U^3D}, \tag{3.2}$$

where we have suppressed the notation U^3D.

Define the generating functions for $\{C_n b_n\}_{n=0}^{\infty}$, $\{C_n g_n\}_{n=0}^{\infty}$ and $\{C_n a_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} \mathcal{B}^{U^3D}(t) &= \sum_{n=0}^{\infty} C_n b_n t^n; \\ \mathcal{G}^{U^3D}(t) &= \sum_{n=0}^{\infty} C_n g_n t^n; \\ \mathcal{A}^{U^3D}(t) &= \sum_{n=0}^{\infty} C_n a_n t^n. \end{aligned} \tag{3.3}$$

We will derive a system of three linear equations for $\mathcal{B}^{U^3D}(t)$, $\mathcal{G}^{U^3D}(t)$ and $\mathcal{A}^{U^3D}(t)$ and then solve for one of them explicitly. From the definitions of $B_n^{U^3D}$, $G_n^{U^3D}$ and $A_n^{U^3D}$, we have

$$B_n^{U^3D} \equiv 0,\ 0 \le n \le 4;\quad G_n^{U^3D} \equiv 0,\ 0 \le n \le 1;\quad A_n^{U^3D} = 0,\ 0 \le n \le 2.$$

Thus,

$$\begin{aligned} &b_n = 0,\ 0 \le n \le 4;\\ &g_n = 0,\ 0 \le n \le 1;\\ &a_n = 0,\ 0 \le n \le 2. \end{aligned} \tag{3.4}$$

The following proposition is the key to obtaining a set of three linear equations for the generating functions $\mathcal{B}^{U^3D}(t)$, $\mathcal{G}^{U^3D}(t)$ and $\mathcal{A}^{U^3D}(t)$.

Proposition 2. *i.*

$$\begin{aligned} &B_n^{U^3D}|\{\sigma_j = n\} \stackrel{dist}{=} A_{j-1}^{U^3D} + B_{n-j}^{U^3D},\ j \in [n-1],\ n \ge 2;\\ &B_n^{U^3D}|\{\sigma_n = n\} \stackrel{dist}{=} B_{n-1}^{U^3D},\ n \ge 2, \end{aligned} \tag{3.5}$$

where on the right hand side of (3.5), $A_{j-1}^{U^3D}$ *and* $B_{n-j}^{U^3D}$ *are understood to be independent.*

ii.

(3.6)

$$\begin{aligned} &G_n^{U^3D}|\{\sigma_1 = n\} \stackrel{dist}{=} G_{n-1}^{U^3D},\ n \ge 2;\\ &G_n^{U^3D}|\{\sigma_j = n\} \stackrel{dist}{=} \left(A_{j-1}^{U^3D} + G_{n-j}^{U^3D}\right) 1_{\{G_{n-j}^{U^3D} \ne 0\}} + \left(B_{j-1}^{U^2D} + 1\right) 1_{\{G_{n-j}^{U^3D} = 0\}},\\ &j \in \{2, \cdots, n\},\ n \ge 2, \end{aligned}$$

where on the right hand side of (3.6), $A_{j-1}^{U^3D}$ *and* $G_{n-j}^{U^3D}$ *are understood to be independent and* $B_{j-1}^{U^3D}$ *and* $G_{n-j}^{U^3D}$ *are understood to be independent.*

iii.

$$\begin{aligned} &A_n^{U^3D}|\{\sigma_1 = n\} \stackrel{dist}{=} A_{n-1}^{U^3D},\ n \ge 2;\\ &A_n^{U^3D}|\{\sigma_j = n\} \stackrel{dist}{=} \left(A_{j-1}^{U^3D} + A_{n-j}^{U^3D}\right) 1_{\{A_{n-j}^{U^3D} \ne 0\}} + G_{j-1}^{U^3D} 1_{\{A_{n-j}^{U^3D} = 0\}},\\ &j \in \{2, \cdots, n\},\ n \ge 2, \end{aligned} \tag{3.7}$$

where on the right hand side of (3.7), $A_{j-1}^{U^3D}$ *and* $A_{n-j}^{U^3D}$ *are understood to be independent and* $G_{j-1}^{U^3D}$ *and* $A_{n-j}^{U^3D}$ *are understood to be independent.*

Proof. The proof is similar to that of Proposition 1. The first line of (3.5) and the second lines of (3.6) and (3.7) follow from the rather obvious equalities

$$B_n^{U^3D}(\sigma)=B_{n-1}^{U^3D}(\sigma_1\cdots\sigma_{n-1}),\ \text{if } \sigma_n=n;$$
$$G_n^{U^3D}(\sigma)=G_{n-1}^{U^3D}(\sigma_2\cdots\sigma_n),\ \text{if } \sigma_1=n;$$
$$A_n^{U^3D}(\sigma)=A_{n-1}^{U^3D}(\sigma_2\cdots\sigma_n),\ \text{if } \sigma_1=n.$$

Recall the notation $\text{red}(\sigma)$ that was introduced in the paragraph containing (2.5). The first line of (3.5) follows from the equality

$$\begin{aligned}&B_n^{U^3D}(\sigma)=A_{j-1}^{U^3D}(\text{red}(\sigma_1\cdots\sigma_{j-1}))+B_{n-j}^{U^3D}(\sigma_{j+1}\cdots\sigma_n),\ \text{if } \sigma_j=n,\\&\text{for } j\in[n-1], n\ge 2,\end{aligned}\tag{3.8}$$

along with the fact that under the conditioned measure $P_n^{\text{av}(132)}|\{\sigma_j=n\}$, the permutation $\text{red}(\sigma_1\cdots\sigma_{j-1})\in S_{j-1}$ has the distribution $P_{j-1}^{\text{av}(132)}$, the permutation $\sigma_{j+1}\cdots\sigma_n\in S_{n-j}$ has the distribution $P_{n-j}^{\text{av}(132)}$, and these two permutations are independent. The explanation for (3.8) is essentially the same as the explanation for (2.9). Generically, $B_n^{U^3D}(\sigma)$ is the sum of two terms. One of the terms is $A_{j-1}^{U^3D}(\text{red}(\sigma_1\cdots\sigma_{j-1}))$, which counts the number of full sets of U^3D and then adds one for an extra UU. This extra UU, along with $\sigma_j=n$ and σ_{j+1} supply an additional full set U^3D which was counted by $A_{j-1}^{U^3D}(\text{red}(\sigma_1\cdots\sigma_{j-1}))$ (via the adding one for the extra UU). The other term is $B_{n-j}^{U^3D}(\sigma_{j+1}\cdots\sigma_n)$, which counts the remaining sets of U^3D.

The second line of (3.6) follows from the equality

$$\begin{aligned}&G_n^{U^3D}(\sigma)=\Big(A_{j-1}^{U^3D}(\text{red}(\sigma_1\cdots\sigma_{j-1}))+G_{n-j}^{U^3D}(\sigma_{j+1}\cdots\sigma_n)\Big)1_{G_{n-j}^{U^3D}(\sigma_{j+1}\cdots\sigma_n)\neq 0}+\\&\Big(B_{j-1}^{U^2D}(\text{red}(\sigma_1\cdots\sigma_{j-1}))+1\Big)1_{G_{n-j}^{U^3D}(\sigma_{j+1}\cdots\sigma_n)=0},\\&\text{if } \sigma_j=n,\ \text{for } j\in[n-1], n\ge 2,\end{aligned}\tag{3.9}$$

along with the fact that under the conditioned measure $P_n^{\text{av}(132)}|\{\sigma_j=n\}$, the permutation $\text{red}(\sigma_1\cdots\sigma_{j-1})\in S_{j-1}$ has the distribution $P_{j-1}^{\text{av}(132)}$, the permutation $\sigma_{j+1}\cdots\sigma_n\in S_{n-j}$ has the distribution $P_{n-j}^{\text{av}(132)}$, and these two permutations are independent. The explanation for (3.9) in the case that $G_{n-j}^{U^3D}(\sigma_{j+1}\cdots\sigma_n)\neq 0$ is similar to the reasoning for (3.8). We explain (3.9) in the case that $G_{n-j}^{U^3D}(\sigma_{j+1}\cdots\sigma_n)=0$ with an example, the same example used to explain (2.10) in the case that $A^{U^2D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=0$. Consider $\sigma=435786921$. So $n=9$ and $j=7$. We have $G^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=G^{U^3D}_2(21)=0$. We have $B^{U^3D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))=B^{U^3D}_6(\text{red}(435786))=B^{U^3D}_6(213564)=1$ because the subsequence 23564 (as well as 13564) corresponds to U^3D. And we have $G^{U^3D}_n(\sigma)=G^{U^3D}_9(435786921)=2$ because the subsequence 457869 (as well as 357869) corresponds to U^3DU. (Note that $A^{U^3D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))=A^{U^3D}_6(\text{red}(435786))=A^{U^3D}_6(213564)=1$ because the subsequence 235 (as well as several others) corresponds to UU. Thus, when $G^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=0$, it is not true in general that $G^{U^3D}_n(\sigma)=A^{U^3D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))+G^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)$.)

The second line of (3.7) follows from the equality

(3.10)
$$A^{U^3D}_n(\sigma)=\Big(A^{U^3D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))+A^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)\Big)\,1_{A^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)\neq 0}+\Big(G^{U^3D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))\Big)\,1_{A^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=0},$$
$$\text{if } \sigma_j=n,\ \text{for } j\in[n-1], n\geq 2,$$

along with the fact that under the conditioned measure $P^{\text{av}(132)}_n|\{\sigma_j=n\}$, the permutation $\text{red}(\sigma_1\cdots\sigma_{j-1})\in S_{j-1}$ has the distribution $P^{\text{av}(132)}_{j-1}$, the permutation $\sigma_{j+1}\cdots\sigma_n\in S_{n-j}$ has the distribution $P^{\text{av}(132)}_{n-j}$, and these two permutations are independent. The explanation for (3.10) in the case that $A^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)\neq 0$ is similar to the reasoning for (3.8). We explain (3.10) in the case that $A^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=0$ with an example. Consider $\sigma=786543921$. So $n=9$ and $j=7$. We have $A^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=A^{U^3D}_2(21)=0$. We have $G^{U^3D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))=G^{U^3D}_6(\text{red}(786543))=G^{U^3D}_6(564321)=1$ because the subsequence 56 corresponds to U. And we have $A^{U^3D}_n(\sigma)=A^{U^3D}_9(786543921)=1$ because the subsequence 789 corresponds to UU. (Note that $A^{U^3D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))=A^{U^3D}_6(\text{red}(786543))=A^{U^3D}_6(564321)=0$. Thus, the equality $A^{U^3D}_n(\sigma)=A^{U^3D}_{j-1}(\text{red}(\sigma_1\cdots\sigma_{j-1}))+A^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)$ is not true in general when $A^{U^3D}_{n-j}(\sigma_{j+1}\cdots\sigma_n)=0$.) □

We now use Proposition 2 to derive a system of three linear equations for $\mathcal{B}^{U^3D}(t)$, $\mathcal{G}^{U^3D}(t)$ and $\mathcal{A}^{U^3D}(t)$. Note from (2.7) and (3.5) that the conditional distributions of $B^{U^2D}(t)$ and $B^{U^3D}(t)$ are exactly the same except that $A^{U^2D}(t)$ in (2.7) is replaced by $A^{U^3D}(t)$ in (3.5). Thus, it follows from

(2.14) that

$$\mathcal{B}^{U^3D}(t)=\frac{t\,(C(t)-1)\,\mathcal{A}^{U^3D}(t)}{1-t-tC(t)}. \tag{3.11}$$

We now turn to $\mathcal{G}^{U^3D}(t)$. Note that for $l\in\mathbb{N}$ and $\sigma\in S_l(132)$, $G_l^{U^3D}(\sigma)=0$ only for $\sigma=l\cdots 21$; thus $P_l^{\text{av}(132)}(G_l^{U^3D}=0)=\frac{1}{C_l}$. Using this with (2.5) and (3.6), it follows that

(3.12)

$$g_n=E_n^{\text{av}(132)}G_n^{U^3D}=\sum_{j=1}^n E_n^{\text{av}(132)}(G_n^{U^3D}|\sigma_j=n)P_n^{\text{av}(132)}(\sigma_j=n)=\frac{C_0C_{n-1}}{C_n}g_{n-1}+$$

$$\sum_{j=2}^n\left(a_{j-1}\left(1-\frac{1}{C_{n-j}}\right)+g_{n-j}\right)\frac{C_{j-1}C_{n-j}}{C_n}+\sum_{j=2}^n\frac{b_{j-1}+1}{C_{n-j}}\frac{C_{j-1}C_{n-j}}{C_n}.$$

Multiplying both sides of (3.12) by C_nt^n, summing over n from 2 to ∞ and using (3.4), we obtain

(3.13)

$$\mathcal{G}^{U^3D}(t)=\sum_{n=2}^\infty C_ng_nt^n=$$

$$t\sum_{n=2}^\infty C_{n-1}g_{n-1}t^{n-1}+t\sum_{n=2}^\infty\left(\sum_{j=2}^n a_{j-1}C_{j-1}C_{n-j}\right)t^{n-1}-t\sum_{n=2}^\infty(\sum_{j=2}^n a_{j-1}C_{j-1})t^{n-1}+$$

$$t\sum_{n=2}^\infty\left(\sum_{j=2}^n C_{j-1}g_{n-j}C_{n-j}\right)t^{n-1}+t\sum_{n=2}^\infty\left(\sum_{j=2}^n b_{j-1}C_{j-1}\right)t^{n-1}+t\sum_{n=2}^\infty\left(\sum_{j=2}^n C_{j-1}\right)t^{n-1}.$$

Considerations almost identical to those that led from (2.16) to (2.18) yield

(3.14)

$$\mathcal{G}^{U^3D}(t)(t)=t\Bigg(\mathcal{G}^{U^3D}(t)(t)+\mathcal{A}^{U^3D}(t)(t)C(t)-\frac{\mathcal{A}^{U^3D}(t)(t)}{1-t}+\mathcal{G}^{U^3D}(t)(t)\,(C(t)-1)+$$

$$\frac{\mathcal{B}^{U^3D}(t)}{1-t}+\frac{C(t)-1}{1-t}\Bigg),$$

which we write as

(3.15)

$$\mathcal{G}^{U^3D}(t)(t)=\frac{\left(tC(t)-\frac{t}{1-t}\right)\mathcal{A}^{U^3D}(t)+\frac{t}{1-t}\mathcal{B}^{U^3D}(t)+\frac{t}{1-t}\,(C(t)-1)}{1-tC(t)}.$$

We now turn to $\mathcal{A}^{U^3D}(t)$. We need the following lemma.

Lemma 1.

$$P_l^{av(132)}(A_l^{U^3D}=0)=\frac{2^{l-1}}{C_l},\ l\in\mathbb{N}. \tag{3.16}$$

Remark. The lemma is equivalent to the fact that the number of permutations in S_l that simultaneously avoid 123 and 132 is equal to 2^{l-1}. This can be found for example in [12]. However, we provide an alternative proof because several of its components will be needed afterwards.

Proof. For convenience, define

$$\begin{aligned}&\gamma_l=P_l^{\text{av}(132)}(A_l^{U^3D}=0),\ l\in\mathbb{N};\\&\gamma_0=1.\end{aligned} \tag{3.17}$$

For $\sigma\in S_l$, distributed as $P_l^{\text{av}(132)}$, and conditioned on $\sigma_i=l$, the permutations $\text{red}(\sigma_1\cdots\sigma_{i-1})$ and $\sigma_{i+1}\cdots\sigma_l$ are independent and distributed respectively as $P_{i-1}^{\text{av}(132)}$ and $P_{l-i}^{\text{av}(132)}$. If $\sigma_i=l$, then $A_l^{U^3D}(\sigma)=0$ if and only if $G_{i-1}^{U^3D}(\text{red}(\sigma_1\cdots\sigma_{i-1}))=0$ and $A_{l-i}^{U^3D}(\sigma_{i+1}\cdots\sigma_l)=0$. Now $G_{i-1}^{U^3D}(\text{red}(\sigma_1\cdots\sigma_{i-1}))=0$ if and only if $\text{red}(\sigma_1\cdots\sigma_{i-1})=i-1\cdots 21$. Thus, $P_{i-1}^{\text{av}(132)}\Big(G_{i-1}^{U^3D}(\text{red}(\sigma_1\cdots\sigma_{i-1}))=0\Big)=\frac{1}{C_{i-1}}$. Therefore, we have

$$P_l^{\text{av}(132)}\Big(A_l^{U^3D}(\sigma)=0|\sigma_i=l\Big)=\frac{\gamma_{l-i}}{C_{i-1}}.$$

Consequently,

$$\gamma_l=P_l^{\text{av}(132)}(A_l^{U^3D}=0)=\sum_{i=1}^{l}\frac{C_{i-1}C_{l-i}}{C_l}\frac{\gamma_{l-i}}{C_{i-1}}=\sum_{i=1}^{l}\frac{C_{l-i}\gamma_{l-i}}{C_l},$$

which we write as

$$k_l=\sum_{i=0}^{l-1}k_i,\quad k_i=C_i\gamma_i. \tag{3.18}$$

Multiply both sides of (3.18) by t^l and write the resulting equation as

$$k_lt^l=t\sum_{i=0}^{l-1}k_it^it^{l-1-i}. \tag{3.19}$$

Let $K(t)=\sum_{l=0}^{\infty}k_lt^l$. Summing (3.19) over l from 1 to ∞, one obtains after some algebra

$$K(t)=1+\frac{tK(t)}{1-t},$$

which yields

$$K(t)=\frac{1-t}{1-2t}=1+\frac{t}{1-2t}=1+\sum_{l=1}^{\infty}2^{l-1}t^l. \tag{3.20}$$

Thus, $C_l\gamma_l=k_l=2^{l-1},\ l\ge 1$. Consequently $P_l^{\mathrm{av}(132)}(A_l^{U^3D}=0)=\gamma_l=\frac{2^{l-1}}{C_l}$. □

Using (3.17) with (2.5) and (3.7), it follows that

$$\begin{aligned}
&a_n=E_n^{\mathrm{av}(132)}A_n^{U^3D}=\sum_{j=1}^n E_n^{\mathrm{av}(132)}(A_n^{U^3D}|\sigma_j=n)P_n^{\mathrm{av}(132)}(\sigma_j=n)=\\
&\frac{C_0C_{n-1}}{C_n}a_{n-1}+\sum_{j=2}^n\left(a_{j-1}\left(1-\gamma_{n-j}\right)+a_{n-j}\right)\frac{C_{j-1}C_{n-j}}{C_n}+\\
&\sum_{j=2}^n g_{j-1}\gamma_{n-j}\,\frac{C_{j-1}C_{n-j}}{C_n}.
\end{aligned} \tag{3.21}$$

Multiplying both sides of (3.21) by C_nt^n and summing over n from 2 to ∞, and recalling (3.4), we obtain

$$\begin{aligned}
&\mathcal{A}^{U^3D}(t)=\sum_{n=2}^{\infty}C_na_nt^n=t\sum_{n=2}^{\infty}C_{n-1}a_{n-1}t^{n-1}+t\sum_{n=2}^{\infty}\left(\sum_{j=2}^n C_{j-1}a_{j-1}C_{n-j}\right)t^{n-1}-\\
&t\sum_{n=2}^{\infty}\left(\sum_{j=2}^n C_{j-1}a_{j-1}\gamma_{n-j}C_{n-j}\right)t^{n-1}+t\sum_{n=2}^{\infty}\left(\sum_{j=2}^n C_{j-1}C_{n-j}a_{n-j}\right)t^{n-1}+\\
&t\sum_{n=2}^{\infty}\left(\sum_{j=2}^n C_{j-1}g_{j-1}\gamma_{n-j}C_{n-j}\right)t^{n-1}.
\end{aligned} \tag{3.22}$$

By (3.18), $\gamma_{n-j}C_{n-j} = k_{n-j}$. Using this with (3.20), we have

(3.23)

$$\sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} C_{j-1}a_{j-1}\gamma_{n-j}C_{n-j}\right)t^{n-1} = \sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} C_{j-1}a_{j-1}k_{n-j}\right)t^{n-1} =$$

$$K(t)\mathcal{A}^{U^3D}(t) = \frac{1-t}{1-2t}\mathcal{A}^{U^3D}(t);$$

$$\sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} C_{j-1}g_{j-1}\gamma_{n-j}C_{n-j}\right)t^{n-1} = \sum_{n=2}^{\infty}\left(\sum_{j=2}^{n} C_{j-1}g_{j-1}k_{n-j}\right)t^{n-1} =$$

$$K(t)\mathcal{G}^{U^3D}(t) = \frac{1-t}{1-2t}\mathcal{G}^{U^3D}(t).$$

The two terms on the left hand sides of (3.23) appear on the right hand side of (3.22). The other terms on the right hand side of (3.22) can be treated via straightforward algebraic calculations, similar to what was done in previous calculations. This allows for (3.22) to be written term by term as

$$\mathcal{A}^{U^3D}(t) = t\mathcal{A}^{U^3D}(t) + t\mathcal{A}^{U^3D}(t)C(t) - \frac{1-t}{1-2t}\mathcal{A}^{U^3D}(t)+$$

$$t\mathcal{A}^{U^3D}(t)\left(C(t)-1\right) + \frac{1-t}{1-2t}\mathcal{G}^{U^3D}(t),$$

which yields

$$\mathcal{A}^{U^3D}(t) = \frac{\frac{t(1-t)}{1-2t}\mathcal{G}^{U^3D}(t)}{1-2tC(t)+\frac{t(1-t)}{1-2t}}. \tag{3.24}$$

Now (3.11), (3.15) and (3.24) provide a system of three linear equations for the three generating functions $\mathcal{B}^{U^3D}(t), \mathcal{G}^{U^3D}(t)$ and $\mathcal{A}^{U^3D}(t)$. Since $G_n^{U^3D}(\sigma), A_n^{U^3D}(\sigma) \in \{B_n^{U^3D}(\sigma)-1, B_n^{U^3D}(\sigma), B_n^{U^3D}(\sigma)+1\}$, for all $n \in \mathbb{N}$ and all $\sigma \in S_n$, the leading order asymptotic behavior is the same for $E_n^{\text{av}(132)}B_n^{U^3D}, E_n^{\text{av}(132)}G_n^{U^3D}$ and $E_n^{\text{av}(132)}A_n^{U^3D}$. Thus, it doesn't matter which of the generating functions we solve for. We will solve for $\mathcal{G}^{U^3D}(t)$. We start with (3.15), and replace the term $\mathcal{B}^{U^3D}(t)$ on the right hand side of (3.15) with the right hand side of (3.11). After rearranging some terms, this gives

$$\mathcal{G}^{U^3D}(t) = \frac{t(C(t)-1)}{(1-t)\left(1-tC(t)\right)}+$$
$$\left(tC(t) - \frac{t}{1-t} + \frac{t^2(C(t)-1)}{(1-t)\left(1-t-tC(t)\right)}\right)\frac{1}{1-tC(t)}\mathcal{A}^{U^3D}(t). \tag{3.25}$$

Now we replace $\mathcal{A}^{U^3D}(t)$ on the right hand side of (3.25) with the right hand side of (3.24). This yields an equation in which only the generating function $\mathcal{G}^{U^3D}(t)$ appears. Solving for $\mathcal{G}^{U^3D}(t)$, we obtain

(3.26)

$$\mathcal{G}^{U^3D}(t) = \frac{t(C(t)-1)}{(1-t)\,(1-tC(t))\,(1-d_1(t))}, \text{ where}$$

$$d_1(t) = \left(tC(t) - \frac{t}{1-t} + \frac{t^2(C(t)-1)}{(1-t)(1-t-tC(t))}\right)\left(\frac{1}{1-tC(t)}\right)\left(\frac{t(1-t)}{1-2t}\right)\times \left(\frac{1}{1-2tC(t)+\frac{t(1-t)}{1-2t}}\right).$$

4. Completion of the proof of Theorem 2

In (3.26), when we perform the multiplication $(1-t)\,(1-tC(t))\,d_1(t)$, the second of the four factors in $d_1(t)$ will disappear, and the $1-t$ in the denominator of two of the terms in the first factor will also disappear. We obtain

(4.1)

$$(1-t)\,(1-tC(t))\,d_1(t) = \left((1-t)tC(t) - t + \frac{t^2(C(t)-1)}{1-t-tC(t)}\right)\left(\frac{t(1-t)}{1-2t}\right)\times \left(\frac{1}{1-2tC(t)+\frac{t(1-t)}{1-2t}}\right).$$

Multiplying the denominators of the second and third factors on the right hand side of (4.1), we have

$$(1-2t)\left(1-2tC(t)+\frac{t(1-t)}{1-2t}\right) = 1-t-t^2-2tC(t)+4t^2C(t). \tag{4.2}$$

Thus, multiplying both the numerator and the denominator on the right hand side of (3.26) by $1-t-t^2-2tC(t)+4t^2C(t)$, and using (4.1) and (4.2), we obtain

$$\mathcal{G}^{U^3D}(t) = \frac{t(C(t)-1)\left(1-t-t^2-2tC(t)+4t^2C(t)\right)}{d_2(t)}, \tag{4.3}$$

where

$$\begin{aligned} d_2(t) &= (1-t)(1-tC(t))\left(1-t-t^2-2tC(t)+4t^2C(t)\right) - \\ &\left((1-t)tC(t) - t + \frac{t^2(C(t)-1)}{1-t-tC(t)}\right) t(1-t). \end{aligned} \tag{4.4}$$

Multiplying the numerator and the denominator on the right hand side of (4.3) by $1-t-tC(t)$, and using (4.4), we obtain

$$\mathcal{G}^{U^3D}(t) = \frac{n(t)}{d(t)}, \tag{4.5}$$

where

(4.6)

$$n(t) = t(C(t)-1)\left(1-t-t^2-2tC(t)+4t^2C(t)\right)(1-t-tC(t));$$

$$d(t) = (1-t)(1-tC(t))\left(1-t-t^2-2tC(t)+4t^2C(t)\right)(1-t-tC(t)) -$$
$$\Big(((1-t)tC(t)-t)(1-t-tC(t))+t^2(C(t)-1)\Big)t(1-t).$$

Grouping powers of $C(t)$, we have

$$\begin{aligned} n(t) &= A_3(t)C^3(t)+A_2(t)C^2(t)+A_1(t)C(t)+A_0(t);\\ d(t) &= B_3(t)C^3(t)+B_2(t)C^2(t)+B_1(t)C(t)+B_0(t), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} &A_3(t) = 2t^3-4t^2;\quad A_2(t) = t^4+5t^3-3t^2;\\ &A_1(t) = 4t^4-7t^3+t^2+t;\quad A_0(t) = -t^4+2t^2-t;\\ &B_3(t) = -4t^5+6t^4-2t^3;\quad B_2(t) = -2t^5+12t^4-15t^3+5t^2;\\ &B_1(t) = 2t^5+t^4-11t^3+12t^2-4t;\quad B_0(t) = -t^4+3t^2-3t+1. \end{aligned} \tag{4.8}$$

Recalling the formula for $C(t)$ in (2.4), we have

$$\begin{aligned} C^2(t) &= \frac{1-2t-\sqrt{1-4t}}{2t^2};\\ C^3(t) &= \frac{1-3t-(1-t)\sqrt{1-4t}}{2t^3}. \end{aligned} \tag{4.9}$$

Letting

$$R := \sqrt{1-4t}$$

and substituting from (2.4) and (4.9) in (4.7), we obtain after a lot of algebra

$$\begin{aligned} n(t) &= \left(-\frac{(1-t)A_3(t)}{2t^3}-\frac{A_2(t)}{2t^2}-\frac{A_1(t)}{2t}\right)R+\\ &\frac{(1-3t)A_3(t)}{2t^3}+\frac{(1-2t)A_2(t)}{2t^2}+\frac{A_1(t)}{2t}+A_0(t);\\ d(t) &= \left(-\frac{(1-t)B_3(t)}{2t^3}-\frac{B_2(t)}{2t^2}-\frac{B_1(t)}{2t}\right)R+\\ &\frac{(1-3t)B_3(t)}{2t^3}+\frac{(1-2t)B_2(t)}{2t^2}+\frac{B_1(t)}{2t}+B_0(t). \end{aligned} \tag{4.10}$$

Using (4.8), one finds that

(4.11)

$$
\begin{aligned}
&-\frac{(1-t)A_3(t)}{2t^3}-\frac{A_2(t)}{2t^2}-\frac{A_1(t)}{2t}=t^2(1-2t);\\
&\frac{(1-3t)A_3(t)}{2t^3}+\frac{(1-2t)A_2(t)}{2t^2}+\frac{A_1(t)}{2t}+A_0(t)=t^3(1-t);\\
&-\frac{(1-t)B_3(t)}{2t^3}-\frac{B_2(t)}{2t^2}-\frac{B_1(t)}{2t}=-t^4-\frac{3}{2}t^3+\frac{9}{2}t^2-\frac{5}{2}t+\frac{1}{2};\\
&\frac{(1-3t)B_3(t)}{2t^3}+\frac{(1-2t)B_2(t)}{2t^2}+\frac{B_1(t)}{2t}+B_0(t)=2t^4-\frac{13}{2}t^3+\frac{15}{2}t^2-\frac{7}{2}t+\frac{1}{2}.
\end{aligned}
$$

From (4.10) and (4.11), we have

(4.12)

$$
\begin{aligned}
&n(t)=t^2(1-2t)R+t^3(1-t);\\
&d(t)=\left(-t^4-\frac{3}{2}t^3+\frac{9}{2}t^2-\frac{5}{2}t+\frac{1}{2}\right)R+2t^4-\frac{13}{2}t^3+\frac{15}{2}t^2-\frac{7}{2}t+\frac{1}{2}.
\end{aligned}
$$

At this point, (4.5) and (4.12) give an explicit formula for $\mathcal{G}^{U^3D}(t)$. Similar to what we did after (2.24) in the proof of the previous theorem, we could try to rewrite this formula exclusively in terms of R and do the singularity analysis with respect to R. But it seems computationally simpler to continue with t and do the singularity analysis with respect to t. Recall that $R=\sqrt{1-4t}$. In order to eliminate the square root in the denominator $d(t)$ in (4.12), we multiply the numerator and denominator by the denominator's conjugate, $-\left(-t^4-\frac{3}{2}t^3+\frac{9}{2}t^2-\frac{5}{2}t+\frac{1}{2}\right)R+2t^4-\frac{13}{2}t^3+\frac{15}{2}t^2-\frac{7}{2}t+\frac{1}{2}$. Calling the resulting numerator and denominator by $\bar n(t)$ and $\bar d(t)$, this yields

$$
\begin{aligned}
&\bar n(t)=\left(-t^6-\frac{9}{2}t^5+21t^4-\frac{57}{2}t^3+\frac{35}{2}t^2-5t+\frac{1}{2}\right)\sqrt{1-4t}+\\
&6t^6+\frac{29}{2}t^5-58t^4+\frac{119}{2}t^3-\frac{55}{2}t^2+6t-\frac{1}{2};\\
&\bar d(t)=4t^7+15t^6-56t^5+45t^4+4t^3-19t^2+8t-1.
\end{aligned}
\tag{4.13}
$$

The new denominator factors as

$$
\begin{aligned}
&\bar d(t)=4t^7+15t^6-56t^5+45t^4+4t^3-19t^2+8t-1=\\
&(1-4t)(1-t)^2\left(-t^4-6t^3+2t^2+2t-1\right).
\end{aligned}
\tag{4.14}
$$

The two polynomials in the new numerator $\bar{n}(t)$ factor as

$$\begin{aligned} & -t^6 - \frac{9}{2}t^5 + 21t^4 - \frac{57}{2}t^3 + \frac{35}{2}t^2 - 5t + \frac{1}{2} = \\ & (1-t)^2\left(-t^4 - \frac{13}{2}t^3 + 9t^2 - 4t + \frac{1}{2}\right); \\ & 6t^6 + \frac{29}{2}t^5 - 58t^4 + \frac{119}{2}t^3 - \frac{55}{2}t^2 + 6t - \frac{1}{2} = \\ & (1-4t)(1-t)\left(\frac{3}{2}t^4 + \frac{11}{2}t^3 - 8t^2 + \frac{7}{2}t - \frac{1}{2}\right). \end{aligned} \tag{4.15}$$

From (4.13)-(4.15) and (4.5), we conclude that

$$\begin{aligned} \mathcal{G}^{U^3D}(t) = & \frac{-t^4 - \frac{13}{2}t^3 + 9t^2 - 4t + \frac{1}{2}}{-t^4 - 6t^3 + 2t^2 + 2t - 1}(1-4t)^{-\frac{1}{2}} + \\ & \frac{\frac{3}{2}t^4 + \frac{11}{2}t^3 - 8t^2 + \frac{7}{2}t - \frac{1}{2}}{(1-t)\left(-t^4 - 6t^3 + 2t^2 + 2t - 1\right)}. \end{aligned} \tag{4.16}$$

The smallest absolute value among the roots of $-t^4 - 6t^3 + 2t^2 + 2t - 1$ is larger than $\frac{1}{4}$; thus, $\frac{1}{-t^4-6t^3+2t^2+2t-1}$ and $\frac{1}{(1-t)(-t^4-6t^3+2t^2+2t-1)}$ are analytic in a ball centered at the origin of radius larger than $\frac{1}{4}$. From the transfer theorem [5, Theorem VI.3, p. 390, Example VI.2, p. 395] and (2.26), it follows that if $g(t)$ is analytic in a disk, centered at the origin, of radius larger than $\frac{1}{4}$, then

$$[t^n]g(t)\,(1-4t)^{-\alpha} = g(\frac{1}{4})4^n\frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right), \text{ for } \alpha \in \mathbb{C} - \mathbb{Z}_{\le 0}. \tag{4.17}$$

Thus, applying (4.17) in the case $\alpha = \frac{1}{2}$ with $g(t) = \frac{1}{-t^4-6t^3+2t^2+2t-1}$ and with $g(t) = \frac{1}{(1-t)(-t^4-6t^3+2t^2+2t-1)}$, it follows from (4.16) that the leading order asymptotic contribution to $[t^n]\mathcal{G}^{U^3D}(t)$ comes from the term $\frac{-t^4-\frac{13}{2}t^3+9t^2-4t+\frac{1}{2}}{-t^4-6t^3+2t^2+2t-1}(1-4t)^{-\frac{1}{2}}$. Since $\frac{1}{-t^4-6t^3+2t^2+2t-1}|_{t=\frac{1}{4}} = -\frac{256}{121}$, we conclude from (4.17) that

$$\begin{aligned} & [t^n]\mathcal{G}^{U^3D}(t) \sim \frac{256}{121}4^n\frac{n^{-\frac{1}{2}}}{\sqrt{\pi}}\left(4^{-4} + \frac{13}{2}\cdot 4^{-3} - 9\cdot 4^{-2} + 4\cdot 4^{-1} - \frac{1}{2}\right) = \\ & \frac{256}{121}4^n\frac{n^{-\frac{1}{2}}}{\sqrt{\pi}}\frac{11}{256} = \frac{1}{11}4^n\frac{n^{-\frac{1}{2}}}{\sqrt{\pi}}. \end{aligned} \tag{4.18}$$

From (3.3) and (3.2), we have $[t^n]\mathcal{G}^{U^3D}(t) = C_n g_n = C_n E_n^{\text{av}(132)} G_n^{U^3D}$. As previously noted, the Catalan numbers satisfy $C_n \sim 4^n\frac{n^{-\frac{3}{2}}}{\sqrt{\pi}}$. Using these facts with (4.18), we conclude that

$$E_n^{\mathrm{av}(132)} G_n^{U^3D} \sim \frac{1}{11} n. \tag{4.19}$$

Theorem 2 now follows from (4.19), (3.1) and the fact that the leading order asymptotic behavior of $E_n^{\mathrm{av}(132)} B_n^{U^3D}$ and of $E_n^{\mathrm{av}(132)} G_n^{U^3D}$ coincide. □

5. Proof of Corollary 1

Recall that the *reverse* of a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is the permutation $\sigma^{\mathrm{rev}} := \sigma_n \cdots \sigma_1$, and the *complement* of σ is the permutation σ^{com} satisfying $\sigma_i^{\mathrm{com}} = n + 1 - \sigma_i,\ i = 1, \cdots, n$. Let $\sigma^{\text{rev-com}}$ denote the permutation obtained by applying reversal and then complementation to σ (or equivalently, vice versa). Since $132^{\mathrm{rev}} = 231$, $132^{\mathrm{comp}} = 312$ and $132^{\text{rev-com}} = 213$, if follows that the three operations, reversal, complementation and reversal-complementation, are bijections from $S_n(132)$ to $S_n(\tau)$, with $\tau = 231$ in the case of reversal, $\tau = 312$ in the case of complementation and $\tau = 213$ in the case of reversal-complementation. From these facts and Theorems 1 and 2, the corollary follows immediately for $\tau \in \{231, 312, 213\}$.

There is a well-known explicit bijection between $S_n(132)$ and $S_n(123)$ [15, 3]. Recall that an entry $j \in [n]$ of a permutation $\sigma \in S_n$ is called a *left-to-right minimum* if $\sigma_j = \min\{\sigma_i : 1 \le i \le j\}$. For a permutation $\sigma \in S_n(132)$, let $\{i_j\}_{j=1}^k$ denote its left-to-right minima. Then necessarily the entries of σ that appear from left to right between σ_{i_j} and $\sigma_{i_{j+1}}$ (or after σ_{i_k} up through the final term in the permutation) are increasing, with each entry being the smallest number remaining that is larger than its predecessor. (In particular, the leftmost such entry is the smallest remaining number larger than σ_{i_j}.) The bijection between $S_n(132)$ and $S_n(123)$ preserves the set of left-to-right minima, and then rearranges all of the other entries in descending order from left to right. Note that the values of the permutation at the left-to-right minima form a decreasing sequence, and these other rearranged entries also form a decreasing sequence; thus the permutation obtained is the union of two decreasing sequences, which is equivalent to its being 123-avoiding. One can check easily that for the pattern UUD (or $UUUD$), there is either no copy or one copy of the pattern between two consecutive left-to-right minima in the permutation $\sigma \in S_n(132)$, and that the same number of copies of UDD (or $UDDD$) appear between those two consecutive left-to-right minima in the 123-avoiding permutation obtained from σ via the above described bijection. This proves the corollary for $\tau = 123$. Applying reversal to 123 proves the corollary for $\tau = 321$. □

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