THE INVERSION STATISTIC IN DERANGEMENTS AND IN OTHER PERMUTATIONS WITH A PRESCRIBED NUMBER OF FIXED POINTS

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ABSTRACT. We study how the inversion statistic is influenced by fixed points in a permutation. For each $n \in \mathbb{N}$, and each $k \in \{0, 1, \cdots, n\}$, let $P_n^{(k)}$ denote the uniform probability measure on the set of permutations in S_n with exactly k fixed points. We obtain an exact formula for the expected number of inversions under the measure $P_n^{(k)}$ as well as for $P_n^{(k)}(\sigma_i^{-1} < \sigma_j^{-1})$, for $1 \le i < j \le n$, the $P_n^{(k)}$-probability that the number i precedes the number j. In particular, up to a super-exponentially small correction as $n \to \infty$, the expected number of inversions in a random derangement $(k = 0)$ is $\frac{1}{6}n + \frac{1}{12}$ more than the value $\frac{n(n-1)}{4}$ that one obtains for a uniformly random general permutation in S_n. On the other hand, up to a super-exponentially small correction, for $k \ge 2$, the expected number of inversions in a random permutation with k fixed points is $\frac{k-1}{6}n + \frac{k^2-k-1}{12}$ less than $\frac{n(n-1)}{4}$. In the borderline case, $k = 1$, up to a super-exponentially small correction, the expected number of inversions in a random permutation with one fixed point is $\frac{1}{12}$ more than $\frac{n(n-1)}{4}$. We can also let k and the pair i, j depend on n. As corollaries of the theorems we obtain the asymptotic behavior of the expected number of inversions under $P_n^{(k_n)}$ when $k_n \to \infty$, for various regimes of k_n, and the asymptotic behavior of $P_n^{(k_n)}(\sigma_{i_n}^{-1} < \sigma_{j_n}^{-1})$, for various regimes of k_n and of $j_n - i_n$. The proofs make strategic and perhaps novel use of the Chinese restaurant construction for a uniformly random permutation.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we study how the inversion statistic is influenced by fixed points in a permutation. Let P_n denote the uniform probability measure on the set S_n of permutations of $[n] = \{1, \cdots, n\}$, and denote the expectation with respect to P_n by E_n. We write $\sigma \in S_n$ in one-line notation as $\sigma = \sigma_1 \cdots \sigma_n$. Recall that σ is a *derangement* if it has no fixed points; that is, if $\sigma_i \neq i$, for all $i \in [n]$. Let $D_n \subset S_n$

denote the set of derangements in S_n. As is well-known [1, 4],

$$(1.1) \qquad P_n(D_n) = \sum_{l=0}^{n} \frac{(-1)^l}{l!}; \quad \lim_{n\to\infty} P_n(D_n) = e^{-1}.$$

Let $I_n(\sigma)$ denote the number of inversions in σ; that is,

$$I_n(\sigma) = \sum_{1 \leq i < j \leq n} 1_{\{\sigma_j < \sigma_i\}} = \sum_{1 \leq i < j \leq n} 1_{\{\sigma_j^{-1} < \sigma_i^{-1}\}}.$$

(Note that σ_i^{-1} is the position of the number i in the permutation σ.) By symmetry, one has $E_n I_n = \frac{n(n-1)}{4}$. Let $I_{n;<j}(\sigma) = \sum_{1 \leq i < j} 1_{\{\sigma_j^{-1} < \sigma_i^{-1}\}}$ denote the number of inversions in σ that involve the number j and a number less than j. Of course, $I_n = \sum_{j=2}^{n} I_{n;<j}$. Under the uniform distribution, it is clear that $I_{n;<j}$ is distributed uniformly on $\{0, 1, \cdots, j-1\}$. Furthermore, it is known that under the uniform distribution the random variables $\{I_{n;<j}\}_{j=2}^{n}$ are independent [3, 5]. Thus, I_n can be represented as the sum of independent random variables. This leads easily to a weak law of large numbers and a central limit theorem for the inversions statistic [3]. One has

$$(1.2) \qquad \lim_{n\to\infty} P_n\left((1-\epsilon)\frac{n^2}{4} \leq I_n \leq (1+\epsilon)\frac{n^2}{4}\right) = 1, \text{ for all } \epsilon > 0,$$

and $n^{-\frac{3}{2}}(I_n - \frac{n^2}{4}) \overset{\text{dist}}{\to} N(0, \frac{1}{36})$.

Denote the uniform probability measure on D_n by $P_n^{(0)}$; that is,

$$(1.3) \qquad P_n^{(0)}(A) = \frac{P_n(A \cap D_n)}{P_n(D_n)}, \ A \subset S_n,$$

and let $E_n^{(0)}$ denote the expectation with respect to $P_n^{(0)}$. From (1.1) and (1.3), it follows that $P_n^{(0)}$ is asymptotically absolutely continuous with respect to P_n, in the sense that if a sequence of events $\{A_n\}_{n=1}^{\infty}$, with $A_n \subset S_n$, satisfies $\lim_{n\to\infty} P_n(A_n) = 0$, then also $\lim_{n\to\infty} P_n^{(0)}(A_n) = 0$. In light of this, it follows that the law of large numbers also holds for I_n under the measure $P_n^{(0)}$; that is, (1.2) also holds with P_n replaced by $P_n^{(0)}$. From this and the fact that $\max_{\sigma \in S_n} I_n(\sigma) = O(n^2)$, if follows that $E_n^{(0)} I_n \sim \frac{n^2}{4}$.

What can be said about lower order terms in $E_n^{(0)} I_n$? Intuitively, it seems that $E_n^{(0)} I_n$ is larger than $E_n I_n = \frac{n(n-1)}{4}$. Indeed, by symmetry considerations, for any $i \in [n]$, one has $P_n^{(0)}(\sigma_i^{-1} = l) = \frac{1}{n-1}$, $l \in [n] - \{i\}$. Thus under $P_n^{(0)}$, for $1 \leq i < j \leq n$, the random variable σ_i^{-1} strictly stochastically dominates the random variable σ_j^{-1}; that is, $P_n^{(0)}(\sigma_i^{-1} \geq a) \geq P_n^{(0)}(\sigma_j^{-1} \geq a)$, for all $a \in \mathbb{R}$, and with strict inequality for at least one choice of a. On the other hand, under P_n, the random variables σ_i^{-1} and σ_j^{-1} have the same distribution; namely they are

uniformly distributed on $[n]$. This is only intuition and not a proof because σ_i^{-1} and σ_j^{-1} are not independent under $P_n^{(0)}$ or under P_n. We will calculate $E_n^{(0)}I_n$ and $P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1})$ explicitly, allowing one to see how much greater $E_n^{(0)}I_n$ is than E_nI_n and how much smaller $P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1})$ is than $P_n(\sigma_i^{-1} < \sigma_j^{-1})$.

We now turn to permutations with a prescribed non-zero number of fixed points. For $k \in \mathbb{N}$, let $D_{n;k}$ denote the set of permutations in S_n with exactly k fixed points; that is

$$D_{n;k} = \{\sigma \in S_n : |\{i \in [n] : \sigma(i) = i\}| = k\}.$$

For convenience, define $D_{n;0} = D_n$. It is well-known [1, 2] that under the uniform measure P_n,

$$(1.4) \qquad \lim_{n \to \infty} P_n(D_{n;k}) = \frac{e^{-1}}{k!}, \ k = 0, 1, \cdots ;$$

equivalently, under P_n, the random variable that counts the number of fixed points converges in distribution to the Poisson distribution with parameter one. Let $P_n^{(k)}$ denote the uniform probability measure on $D_{n;k}$; that is,

$$P_n^{(k)}(A) = \frac{P_n(A \cap D_{n;k})}{P_n(D_{n;k})}, \ A \subset S_n,$$

and let $E_n^{(k)}$ denote the expectation with respect to $P_n^{(k)}$. From (1.4) and the reasoning above in the case $k = 0$, it follows that (1.2) also holds with P_n replaced by $P_n^{(k)}$. Thus, as with the case $k = 0$, we have $E_n^{(k)}I_n \sim \frac{n^2}{4}$. Which do we expect to be larger, $E_n^{(k)}I_n$ or $E_nI_n = \frac{n(n-1)}{4}$? It is instructive to consider the extreme case in which $k = n$. Note that $P_n^{(n)}$ is the δ-measure on the identity permutation; thus, $E_n^{(n)}I_n = 0$. From this fact and the above intuition concerning the case $k = 0$, it is natural to suspect that there is a threshold value of k (perhaps depending on n), so that for k below the threshold, $E_n^{(k)}I_n > E_nI_n$, and for k above the threshold, $E_n^{(k)}I_n < E_nI_n$. Since under the uniform measure, the expected number of fixed points is easily seen to be equal to one for all n, the candidate $k = 1$ is an intuitive choice for the threshold, at least for sufficiently large n. We will calculate $E_n^{(k)}I_n$ explicitly. In particular, we will see that the above noted threshold is indeed $k = 1$, and we will see what happens at the threshold $k = 1$.

The first theorem below concerns derangements and the second one treats permutations with a prescribed non-zero number of fixed points.

Theorem 1. *Let $n \geq 3$.*

i.

$$E_n^{(0)} I_n = \frac{n(n-1)}{4} + \frac{1}{6}n + \frac{1}{12} + \frac{\frac{(-1)^n}{n!}}{\sum_{l=0}^n \frac{(-1)^l}{l!}} \left(\frac{n-1}{12} \right) =$$

(1.5)

$$\frac{n(n-1)}{4} + \frac{1}{6}n + \frac{1}{12} + O\left(\frac{1}{(n-1)!} \right).$$

ii. Let $1 \leq i < j \leq n$. Then

(1.6)

$$P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1}) = \frac{1}{2} + \frac{\left(1 - 2(j-i)\right)n + 2(j-i)}{2n(n-1)(n-2)} + \frac{\frac{(-1)^n}{n!}}{\sum_{l=0}^n \frac{(-1)^l}{l!}} \frac{2(j-i) - n}{2n(n-2)} =$$

$$\frac{1}{2} + \frac{\left(1 - 2(j-i)\right)n + 2(j-i)}{2n(n-1)(n-2)} + O\left(\frac{1}{(n+1)!} \right).$$

Theorem 2. *Let $n \geq 3$ and let $k \in \{1, 2, \cdots, n\} - \{n-1\}$.*

i.

$$E_n^{(k)} I_n = \frac{n(n-1)}{4} - \frac{k-1}{6}n - \frac{k^2 - k - 1}{12} + \frac{\frac{(-1)^{n-k}}{(n-k)!}}{\sum_{l=0}^{n-k} \frac{(-1)^l}{l!}} \left(\frac{n-k-1}{12} \right) =$$

(1.7)

$$\frac{n(n-1)}{4} - \frac{k-1}{6}n - \frac{k^2 - k - 1}{12} + O\left(\frac{1}{(n-k-1)!} \right).$$

ii. Let $1 \leq i < j \leq n$. Then

(1.8)
$$P_n^{(k)}(\sigma_i^{-1} < \sigma_j^{-1}) =$$

$$\frac{1}{2} + \frac{1}{2n(n-1)(n-2)} \Big(\left(2(k-1)(j-i) + k^2 - 3k + 1 \right) n - 2(k^2 - k - 1)(j-i) \Big) +$$

$$O\left(\frac{1}{(n-k)!} \right).$$

Remark. The precise formula for the term $O\left(\frac{1}{(n-k)!} \right)$ in (1.8) can be found at the end of the paper in (3.32).

Remark. From part (i) of Theorem 1, one sees that up to a super-exponentially small correction as $n \to \infty$, the expected number of inversions in the case of a random derangement is $\frac{1}{6}n + \frac{1}{12}$ more than it is for a uniformly random permutation. On the other hand, from part (i) of Theorem 2, one sees that up to a super-exponentially small correction, for $k \geq 2$, the expected number of inversions in a random permutation with k fixed points is $\frac{k-1}{6}n + \frac{k^2 - k - 1}{12}$ less than it is for a uniformly random permutation. In particular, in both the case of derangements ($k = 0$) and in the case $k \geq 2$, this difference grows linearly in n. Now consider the

borderline case, $k = 1$. As we noted above, for all n, the expected number of fixed points in a uniformly random permutation is one. Theorem 2 shows that for all n, up to a super-exponentially small correction, the expected number of inversions in a random permutation with one fixed point is $\frac{1}{12}$ more than it is for a uniformly random permutation.

Remark. As we noted above, the random variable that counts the number of fixed points in a uniformly random permutation of size n converges in distribution as $n \to \infty$ to the Poisson distribution with parameter one. Let X be a random variable with this distribution. Then $EX = 1$ and $EX^2 = 2$. If we substitute X for k in the expression $\frac{k-1}{6}n + \frac{k^2-k-1}{12}$ appearing in the formula for $E_n^{(k)}I_n$ in part (i) of Theorem 2 and take the expectation, we obtain $E(\frac{X-1}{6}n + \frac{X^2-X-1}{12}) = 0$.

Note that since Theorem 2 is true for all $k \in \{1, \cdots, n\}$, one can let k depend on n: $k = k_n$ with $\lim_{n\to\infty} k_n = \infty$. This leads to the following corollary which follows by direct calculation.

Corollary 1. *Assume that $k = k_n$ satisfies $\lim_{n\to\infty} k_n = \infty$.*
i. If $k_n = o(n)$, then $E_n^{(k_n)}I_n \sim \frac{n^2}{4}$;
ii. If $k_n \sim cn$, $c \in (0,1)$, then $E_n^{(k_n)}I_n \sim \frac{3-2c-c^2}{12}n^2$;
iii. If $k_n = n - l_n$, where $l_n \geq 2$ and $l_n = o(n)$, then $E_n^{(k_n)}I_n \sim \frac{l_n-1}{3}n$.

In part (ii) of both Theorems 1 and 2 one can let i and j depend on n: $i = i_n, j = j_n$. This leads to the following corollary which follows by direct calculation.

Corollary 2. *Let $1 \leq i_n < j_n \leq n$.*
i. Let $k = 0$ (the case of derangements). Then

$$0 \leq \frac{1}{2} - P_n^{(0)}(\sigma_{i_n}^{-1} < \sigma_{j_n}^{-1}) = \theta(\frac{j_n - i_n}{n^2}).$$

ii. Let $k = k_n$ satisfy $k_n \geq 2$ and $\limsup_{n\to\infty} \frac{k_n}{n} < 1$. Then

$$0 \leq P_n^{(k_n)}(\sigma_{i_n}^{-1} < \sigma_{j_n}^{-1}) - \frac{1}{2} = \theta\left(\frac{k_n(j_n - i_n) \wedge k_n^2}{n^2}\right).$$

iii. Let $k = 1$. Then

$$P_n^{(1)}(\sigma_{i_n}^{-1} < \sigma_{j_n}^{-1}) = \frac{1}{2} + \frac{1}{2n(n-1)(n-2)}\left(-n + 2(j_n - i_n)\right) + O(\frac{1}{(n-1)!}).$$

In particular, for sufficiently large n, $P_n^{(1)}(\sigma_{i_n}^{-1} < \sigma_{j_n}^{-1}) > \frac{1}{2}$ if $j_n - i_n > \frac{n}{2}$, and $P_n^{(1)}(\sigma_{i_n}^{-1} < \sigma_{j_n}^{-1}) < \frac{1}{2}$ if $j_n - i_n < \frac{n}{2}$.

The proofs of the two theorems make strategic and perhaps novel use of the Chinese restaurant construction for a uniformly random permutation. We note that

if one substitutes $k = 0$ in (1.7) and (1.8), these formulas reduce to (1.5) and (1.6) respectively. We have presented the two results separately because derangements are an important class of permutations and because the formulas are considerably simpler for derangements. Furthermore, although the same general method is used to prove the two theorems, there are some technical differences in the proofs, and the calculations are much shorter for Theorem 1. Theorem 1 is proved in section 2 and Theorem 2 is proved in section 3.

We end this section with a description of the standard Chinese restaurant construction. (A strategic alteration of the construction will be used in the proofs.) This construction simultaneously yields a uniformly random permutation Σ_n in S_n, for all n [6, 4]. Furthermore, the construction is consistent in the sense that if one writes out the permutation Σ_n as the product of its cycles and deletes the number n from the cycle in which it appears, then the resulting random permutation of S_{n-1} is equal to Σ_{n-1}.

The construction works as follows. Consider a restaurant with an unlimited number of circular tables, each of which has an unlimited number of seats. Person number 1 sits at a table. Now for $n \geq 1$, suppose that persons number 1 through n have already been seated. Then person number $n+1$ chooses a seat as follows. For each $j \in [n]$, with probability $\frac{1}{n+1}$, person number $n+1$ chooses to sit to the left of person number j. Also, with probability $\frac{1}{n+1}$, person number $n + 1$ chooses to sit at an unoccupied table. Now for each $n \in \mathbb{N}$, the random permutation $\Sigma_n \in S_n$ is defined by $\Sigma_n(i) = j$, if after the first n persons have taken seats, person number j is seated to the left of person number i.

2. Proof of Theorem 1

For the duration of the proof, n, i, j are fixed, with $1 \leq i < j \leq n$. We begin by calculating $P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1})$. This will prove part (ii) and will also be fundamental for the proof of part (i). We implement the Chinese restaurant construction, described at the end of section 1, to build a uniformly random permutation in S_n. However, we make one change. From the construction, it is clear that the n persons can enter in any order we like, as we still obtain a uniformly random permutation in S_n. We let the number j be the last of the n numbers to be used. That is, in the language of the construction, *person j chooses a seat last, after all the other $n-1$ persons with numbers in $[n] - \{j\}$ have already chosen their seats.* We denote by $\Sigma_n = \Sigma_n(1) \cdots \Sigma_n(n)$ the uniformly random permutation in S_n obtained via this construction. Note that after $n - 1$ stages of the construction, a uniformly

random permutation of $[n] - \{j\}$ has been built. We denote this permutation by $\Sigma_{n-1} = \Sigma_{n-1}(1) \cdots \Sigma_{n-1}(j-1)\Sigma_{n-1}(j+1) \cdots \Sigma_{n-1}(n)$. A fixed point for Σ_{n-1} is a number $l \in [n] - \{j\}$ for which $\Sigma_{n-1}(l) = l$. We note for later use that the distribution of the number of fixed points in Σ_{n-1} is the same as it is for a uniformly random permutation of $[n-1]$. We use the generic P to denote probabilities with respect to the above construction.

We now define several events relative to the above construction. We note that we are are reserving the notation D_l for the subset of derangements in $S_l, l \in \mathbb{N}$. Consequently, we denote by $\mathcal{D}_{n-1}$ the event that Σ_{n-1} has no fixed points and by $\mathcal{D}_n$ the event that Σ_n has no fixed points. Denote by $\mathcal{D}_{n-1,1}$ the event that Σ_{n-1} has one fixed point. Then from the construction, it follows that

$$(2.1) \qquad \mathcal{D}_n = (\mathcal{D}_{n-1} \cap \mathcal{D}_n) \cup (\mathcal{D}_{n-1,1} \cap \mathcal{D}_n).$$

Let

$$(2.2) \qquad C_{n,i,j} = \{\Sigma_n^{-1}(i) < \Sigma_n^{-1}(j)\}.$$

Note that $\Sigma_n^{-1}(i)$ denotes the position of i in Σ_n. By (2.1),

$$(2.3) \qquad \mathcal{D}_n \cap C_{n,i,j} = (\mathcal{D}_{n-1} \cap \mathcal{D}_n \cap C_{n,i,j}) \cup (\mathcal{D}_{n-1,1} \cap \mathcal{D}_n \cap C_{n,i,j}).$$

Thus,

$$(2.4) \qquad P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1}) = \frac{P(\mathcal{D}_{n-1} \cap \mathcal{D}_n \cap C_{n,i,j})}{P(\mathcal{D}_n)} + \frac{P(\mathcal{D}_{n-1,1} \cap \mathcal{D}_n \cap C_{n,i,j})}{P(\mathcal{D}_n)}.$$

We now calculate the first term on the right hand side of (2.4). From the construction, it follows that conditioned on $\mathcal{D}_{n-1}$, the random variable $\Sigma_{n-1}^{-1}(i)$ is uniformly distributed on $[n] - \{i, j\}$, and conditioned on $\mathcal{D}_n$, the random variable $\Sigma_n^{-1}(j)$ is uniformly distributed on $[n] - \{j\}$. Furthermore, conditioned on $\mathcal{D}_{n-1} \cap \mathcal{D}_n$, the random variables $\Sigma_{n-1}^{-1}(i)$ and $\Sigma_n^{-1}(j)$ are independent; thus the random vector $\left(\Sigma_{n-1}^{-1}(i), \Sigma_n^{-1}(j)\right)$ is uniformly distributed on $([n] - \{i, j\}) \times ([n] - \{j\})$. Note from the construction that if $\left(\Sigma_{n-1}^{-1}(i), \Sigma_n^{-1}(j)\right) = (l_1, l_2)$ with $l_1 \neq l_2$, then $\Sigma_n^{-1}(i) = l_1$, but if $l_1 = l_2$, then $\Sigma_n^{-1}(i) = j$. Thus, we conclude that

$$P\left(\left(\Sigma_n^{-1}(i), \Sigma_n^{-1}(j)\right) = (l_1, l_2)\middle|\mathcal{D}_{n-1} \cap \mathcal{D}_n\right) =$$

$$(2.5) \qquad \begin{cases} \frac{1}{(n-2)(n-1)}; \; l_1 \neq l_2, \; (l_1, l_2) \in ([n] - \{i, j\}) \times ([n] - \{j\}); \\ \frac{1}{(n-2)(n-1)}, \; l_1 = j, l_2 \in [n] - \{i, j\}. \end{cases}$$

For each $m \in \mathbb{N}$, let d_m denote the number of derangements in S_m. From (1.1), we have

$$(2.6) \qquad d_m = m! \left(\sum_{l=0}^{m} \frac{(-1)^l}{l!} \right).$$

We write

$$(2.7) \qquad P(\mathcal{D}_m) = \frac{d_m}{m!}.$$

From the construction, we have $P(\mathcal{D}_n | \mathcal{D}_{n-1}) = \frac{n-1}{n}$. Using this with (2.7) gives

$$(2.8) \qquad P(\mathcal{D}_{n-1} \cap \mathcal{D}_n) = (n-1)\frac{d_{n-1}}{n!}.$$

From (2.2), (2.5), and (2.8), we conclude that

(2.9)
$$P(\mathcal{D}_{n-1} \cap \mathcal{D}_n \cap C_{n,i,j}) =$$

$$\frac{(n-1)d_{n-1}}{n!} \frac{1}{(n-2)(n-1)} \left(n - j + \sum_{l_1 \in [j-1]-\{i\}} (n-1-l_1) + \sum_{l_1=j+1}^{n} (n-l_1) \right).$$

We now calculate the second term on the right hand side of (2.4). Let $d_{n-1,1}$ denote the number of permutations in S_{n-1} with one fixed point. It follows easily that $d_{n-1,1} = (n-1)d_{n-2}$, where d_{n-2} is as in (2.6). So

$$(2.10) \qquad P(\mathcal{D}_{n-1,1}) = \frac{d_{n-1,1}}{(n-1)!} = \frac{(n-1)d_{n-2}}{(n-1)!} = \frac{d_{n-2}}{(n-2)!}.$$

Conditioned on $\mathcal{D}_{n-1,1}$, the event $\mathcal{D}_n$ occurs if and only if the number j, which enters at stage n, joins the lone singleton existing at stage $n-1$; the probability of this is of course $\frac{1}{n}$. Thus,

$$(2.11) \qquad P(\mathcal{D}_n | \mathcal{D}_{n-1,1}) = \frac{1}{n}.$$

From (2.10) and (2.11), we conclude that

$$(2.12) \qquad P(\mathcal{D}_{n-1,1} \cap \mathcal{D}_n) = \frac{d_{n-2}}{n(n-2)!}.$$

Now consider $P(C_{n,i,j} | \mathcal{D}_{n-1,1} \cap \mathcal{D}_n)$. As we've already noted, for the event $\mathcal{D}_{n-1,1} \cap \mathcal{D}_n$ to occur, the number j must join the lone singleton existing at stage $n-1$. This singleton has equal probability of being any number in $[n] - \{j\}$. In particular, with probability $\frac{1}{n-1}$, the number j will join the number i. In this case $\Sigma_n^{-1}(i) = j$ and $\Sigma_n^{-1}(j) = i$, and thus the event $\{\Sigma_n^{-1}(i) < \Sigma_n^{-1}(j)\}$ does not occur. Conditioned on the singleton not being i, it follows by symmetry that $(\Sigma_n^{-1}(i), \Sigma_n^{-1}(j))$ is distributed uniformly on $([n] - \{i,j\}) \times ([n] - \{i,j\})$. Thus, in

this case, the event $\{\Sigma_n^{-1}(i) < \Sigma_n^{-1}(j)\}$ occurs with probability $\frac{1}{2}$. From this, we conclude that

$$(2.13) \qquad P(C_{n,i,j}|\mathcal{D}_{n-1,1} \cap \mathcal{D}_n) = \frac{1}{2}\frac{n-2}{n-1}.$$

From (2.12) and (2.13), we obtain

$$(2.14) \qquad P(\mathcal{D}_{n-1,1} \cap \mathcal{D}_n \cap C_{n,i,j}) = \frac{1}{2}\frac{n-2}{n-1}\frac{d_{n-2}}{n(n-2)!} = \frac{1}{2}\frac{(n-2)d_{n-2}}{n!}.$$

From (2.4), (2.7), (2.9) and (2.14), we obtain

$$(2.15)$$
$$\frac{d_n}{n!}P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1}) = \frac{1}{2}\frac{(n-2)d_{n-2}}{n!} +$$
$$\frac{(n-1)d_{n-1}}{n!}\frac{1}{(n-2)(n-1)}\left(n-j+\sum_{l_1 \in [j-1]-\{i\}}(n-1-l_1) + \sum_{l_1=j+1}^{n}(n-l_1)\right).$$

Using the fact that $\sum_{l_1\in[j-1]-\{i\}} l_1 = \frac{(j-1)j}{2} - i$ and that $\sum_{l_1=j+1}^{n} l_1 = \frac{n(n+1)}{2} - \frac{j(j+1)}{2}$, we can rewrite (2.15) as

$$(2.16)$$
$$\frac{d_n}{n!}P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1}) = \frac{1}{2}\frac{(n-2)d_{n-2}}{n!} +$$
$$\frac{d_{n-1}}{(n-2)n!}\left((n-1)(j-2) + (n+1)(n-j) - \frac{(j-1)j}{2} + i - \frac{n(n+1)}{2} + \frac{j(j+1)}{2}\right).$$

One has

$$(n-1)(j-2)+(n+1)(n-j)-\frac{(j-1)j}{2}+i-\frac{n(n+1)}{2}+\frac{j(j+1)}{2} = \frac{1}{2}n^2 - \frac{3}{2}n+i-j+2.$$

Thus, from (2.16) we have

$$(2.17)$$
$$\frac{d_n}{n!}P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1}) = \frac{1}{2}\frac{(n-2)d_{n-2}}{n!} + \frac{d_{n-1}}{(n-2)n!}\left(\frac{1}{2}n^2 - \frac{3}{2}n + i - j + 2\right) =$$
$$= \frac{d_{n-2}}{(n-2)!}\left(\frac{1}{2}\frac{n-2}{n(n-1)}\right) + \frac{d_{n-1}}{(n-1)!}\frac{1}{n(n-2)}\left(\frac{1}{2}n^2 - \frac{3}{2}n + i - j + 2\right).$$

From (2.6), we have

$$(2.18) \qquad \frac{\frac{d_{n-1}}{(n-1)!}}{\frac{d_n}{n!}} = 1 - \frac{\frac{(-1)^n}{n!}}{\sum_{l=0}^{n}\frac{(-1)^l}{l!}}, \qquad \frac{\frac{d_{n-2}}{(n-2)!}}{\frac{d_n}{n!}} = 1 - \frac{\frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}}{\sum_{l=0}^{n}\frac{(-1)^l}{l!}}.$$

Also, we have

$$\frac{1}{n(n-2)}\left(\frac{n^2}{2} - \frac{3}{2}n\right) = \frac{1}{2} - \frac{1}{2(n-2)}.$$

Using this with (2.18), we can rewrite (2.17) as

$$
P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1}) = \left(1 - \frac{\frac{(-1)^n}{n!}}{\sum_{l=0}^n \frac{(-1)^l}{l!}}\right)\left(\frac{1}{2} - \frac{1}{2(n-2)} + \frac{2 - (j-i)}{n(n-2)}\right) +
$$

(2.19)

$$
\left(1 - \frac{\frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}}{\sum_{l=0}^n \frac{(-1)^l}{l!}}\right)\frac{n-2}{2n(n-1)}.
$$

After some algebra, which we leave to the reader, (2.19) can be written as given in (1.6). This proves part (ii) of the theorem.

We now turn to part (i) of the theorem. We have

$$
(2.20) \qquad E_n^0 I_n = \frac{n(n-1)}{2} - \sum_{1 \le i < j \le n} P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1}).
$$

Using (1.6), we calculate $\sum_{1 \le i < j \le n} P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1})$. We note that

$$
(2.21) \qquad \sum_{1 \le i < j \le n} i = \frac{1}{6}(n-1)n(n+1); \qquad \sum_{1 \le i < j \le n} j = \frac{1}{3}(n-1)n(n+1).
$$

We first sum over $1 \le i < j \le n$ the terms on the right hand side of the first line of (1.6) that don't involve $\frac{\frac{(-1)^n}{n!}}{\sum_{l=0}^n \frac{(-1)^l}{l!}}$. Using (2.21), we have

(2.22)

$$
\sum_{1 \le i < j \le n} \left(\frac{1}{2} + \frac{\big(1 - 2(j-i)\big)n + 2(j-i)}{2n(n-1)(n-2)}\right) =
$$

$$
\frac{1}{4}n(n-1) - \frac{n}{2n(n-1)(n-2)}\frac{1}{2}n(n-1) - \frac{2(n-1)}{2n(n-1)(n-2)}\frac{1}{6}(n-1)n(n+1) =
$$

$$
\frac{1}{4}n(n-1) - \frac{n}{6} - \frac{1}{12}.
$$

Now we sum over $1 \le i < j \le n$ the term multiplying $\frac{\frac{(-1)^n}{n!}}{\sum_{l=0}^n \frac{(-1)^l}{l!}}$ on the right hand side of the first line of (1.6). Using (2.21), we have

(2.23)

$$
\sum_{1 \le i < j \le n} \frac{2(j-i) - n}{2n(n-2)} = \frac{1}{n(n-2)}\frac{1}{6}(n-1)n(n+1) - \frac{1}{2(n-2)}\frac{1}{2}n(n-1) = \frac{1-n}{12}.
$$

From (1.6), (2.22) and (2.23), we conclude that

$$
(2.24) \quad \sum_{1 \le i < j \le n} P_n^{(0)}(\sigma_i^{-1} < \sigma_j^{-1}) = \frac{1}{4}n(n-1) - \frac{n}{6} - \frac{1}{12} + \frac{\frac{(-1)^n}{n!}}{\sum_{l=0}^n \frac{(-1)^l}{l!}}\left(\frac{1-n}{12}\right).
$$

Now (1.5) follows from (2.20) and (2.24). This completes the proof of part (i)   □

3. Proof of Theorem 2

For the duration of the proof, n, i, j and k are fixed, with $1 \leq i < j \leq n$ and $1 \leq k \leq n$. We implement the Chinese restaurant construction in the same way that it was implemented in the proof of Theorem 1; that is, the number j enters at the final stage. Recall from the first paragraph of the proof of Theorem 1 that the uniformly random permutation in S_n obtained via the construction is denoted by $\Sigma_n = \Sigma_n(1) \cdots \Sigma_n(n)$, and the uniformly random permutation of $[n] - \{j\}$ obtained after the first $n-1$ stages of the construction is denoted by $\Sigma_{n-1} = \Sigma_{n-1}(1) \cdots \Sigma_{n-1}(j-1)\Sigma_{n-1}(j+1) \cdots \Sigma_{n-1}(n)$.

We now define several events relative to the above construction. We note that we are are reserving the notation $D_{n,l}$ for the subset of permutations in S_n with l fixed points, $l \in [n]$. Consequently, we denote by $\mathcal{D}_{n,k}$ the event that Σ_n has k fixed points, and we denote by $\mathcal{D}_{n-1,l}$ the event that Σ_{n-1} has l fixed points, $l \in \{0, 1, \cdots, n-1\}$. From the construction, it follows that

$$(3.1) \qquad \mathcal{D}_{n,k} = (\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k}) \cup (\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k}) \cup (\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k}).$$

(If $k = n$, we understand $\mathcal{D}_{n-1,k}$ to be the empty set and if $k \in \{n-1, n\}$, we understand $\mathcal{D}_{n-1,k+1}$ to be the empty set.)

Let

$$C_{n,i,j} = \{\Sigma_n^{-1}(i) < \Sigma_n^{-1}(j)\},$$

as was defined in (2.2). By (3.1), we have
(3.2)
$$\mathcal{D}_{n,k} \cap C_{n,i,j} =$$

$$(\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}) \cup (\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}) \cup (\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}).$$

We will calculate the probability of each of the three events in the union on the right hand side of (3.2).

We begin with the calculation of $P(\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap C_{n,i,j})$. The event that i is a fixed point at stage $n-1$ is the event $\{\Sigma_{n-1}^{-1}(i) = i\}$. We write

$$(3.3) \quad \begin{aligned} &\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} = \\ &\left(\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\}\right) \cup \left(\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\}\right). \end{aligned}$$

Let $d_{m,l}$ denote the number of permutations in S_m with l fixed points. Then $P(\mathcal{D}_{n-1,k-1}) = \frac{d_{n-1,k-1}}{(n-1)!}$. Given $\mathcal{D}_{n-1,k-1}$, the probability that $\Sigma_{n-1}^{-1}(i) = i$ is $\frac{k-1}{n-1}$. Given $\mathcal{D}_{n-1,k-1}$, the event $\mathcal{D}_{n,k}$ will occur if and only if j enters at stage n as a fixed point; that is, if and only if $\Sigma_n^{-1}(j) = j$. The probability of this is $\frac{1}{n}$. On the

event, $\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\}$, one has $\Sigma_n^{-1}(i) = i$ and $\Sigma_n^{-1}(j) = j$, and thus the event $C_{n,i,j}$ occurs automatically. From the above facts we obtain

$$(3.4) \qquad P\left(\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\} \cap C_{n,i,j}\right) = \frac{d_{n-1,k-1}}{(n-1)!} \frac{k-1}{n-1} \frac{1}{n}.$$

Given $\mathcal{D}_{n-1,k-1}$, the probability that $\Sigma_{n-1}^{-1}(i) \neq i$ is $\frac{n-k}{n-1}$. Given the event $\mathcal{D}_{n-1,k-1} \cap$ $\mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\}$, $\Sigma_n^{-1}(i)$ is equally distributed over $[n] - \{i,j\}$. Thus, given this event, the event $C_{n,i,j}$ occurs with probability $\frac{j-2}{n-2}$. From these facts we obtain

$$(3.5) \quad P\left(\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\} \cap C_{n,i,j}\right) = \frac{d_{n-1,k-1}}{(n-1)!} \frac{n-k}{n-1} \frac{1}{n} \frac{j-2}{n-2}.$$

From (3.3)–(3.5), we conclude that

$$(3.6) \quad P(\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}) = \frac{d_{n-1,k-1}}{(n-1)!} \left(\frac{k-1}{n(n-1)} + \frac{(n-k)(j-2)}{n(n-1)(n-2)}\right).$$

We now turn to the calculation of $P(\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap C_{n,i,j})$. Similar to what we did in (3.3), we write

$$(3.7) \quad \begin{aligned} &\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} = \\ &\left(\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\}\right) \cup \left(\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\}\right). \end{aligned}$$

We have $P(\mathcal{D}_{n-1,k}) = \frac{d_{n-1,k}}{(n-1)!}$. Given $\mathcal{D}_{n-1,k}$, the probability that $\Sigma_{n-1}^{-1}(i) = i$ is $\frac{k}{n-1}$. Given $\mathcal{D}_{n-1,k}$, the event $\mathcal{D}_{n,k}$ will occur if and only if person j entering at stage n sits at a table that already has at least two people. (If person j starts a new table, then Σ_n will have $k+1$ fixed points, and if person j sits at a table that has one person, then Σ_n will have $k-1$ fixed points.) The probability of this is $\frac{n-k-1}{n}$. Given the event $\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\}$, $\Sigma_n^{-1}(j)$ is uniformly distributed over $[n] - \{i,j\}$; thus given this event, the probability that $C_{n,i,j}$ occurs is $\frac{n-i-1}{n-2}$. From these facts, we obtain

$$(3.8)$$
$$P\left(\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\} \cap C_{n,i,j}\right) = \frac{d_{n-1,k}}{(n-1)!} \frac{k}{n-1} \frac{n-k-1}{n} \frac{n-i-1}{n-2}.$$

We now calculate $P\left(\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\} \cap C_{n,i,j}\right)$. Given $\mathcal{D}_{n-1,k}$, the probability that $\Sigma_{n-1}^{-1}(i) \neq i$ is $\frac{n-k-1}{n-1}$. As noted above, given $\mathcal{D}_{n-1,k}$, the event $\mathcal{D}_{n,k}$ will occur if and only if person j entering at stage n sits at a table that already has at least two people, and the probability of this is $\frac{n-k-1}{n}$. We now need to calculate the probability of $C_{n,i,j}$, given $\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\}$. We need to be somewhat careful here. Conditioned on the above event, $\Sigma_{n-1}^{-1}(i)$ is uniformly distributed over $[n] - \{i,j\}$. Conditioned on the above event, $\Sigma_n^{-1}(j)$ is uniformly distributed over the $n - k - 1$ numbers that are not fixed points for Σ_{n-1}. These

$n - k - 1$ numbers include i. The other $n - k - 2$ such numbers are uniformly distributed over all the $(n - k - 2)$-tuples in $[n] - \{i, j\}$. From this, it follows that

(3.9)
$$P(\Sigma_{n-1}^{-1}(i) = l_1, \Sigma_n^{-1}(j) = i) = \frac{1}{n-2} \frac{1}{n-k-1}; \ l_1 \in [n] - \{i, j\};$$

$$P(\Sigma_{n-1}^{-1}(i) = l_1, \Sigma_n^{-1}(j) = l_1) = \frac{1}{n-2} \frac{1}{n-k-1}, \ l_1 \in [n] - \{i, j\};$$

$$P(\Sigma_{n-1}^{-1}(i) = l_1, \Sigma_n^{-1}(j) = l_2) = \frac{1}{n-2} \frac{n-k-3}{n-k-1} \frac{1}{n-3}, \ l_1 \neq l_2, \ l_1, l_2 \in [n] - \{i, j\}.$$

If $\{\Sigma_{n-1}^{-1}(i) = l_1, \Sigma_n^{-1}(j) = i\}$ occurs with $l_1 \in [n] - \{i, j\}$, then also $\Sigma_n^{-1}(i) = l_1$, and if $\{\Sigma_{n-1}^{-1}(i) = l_1, \Sigma_n^{-1}(j) = l_2)\}$ occurs with $l_1 \neq l_2, \ l_1, l_2 \in [n] - \{i, j\}$, then also $\Sigma_n^{-1}(i) = l_1$. However, if $\{\Sigma_{n-1}^{-1}(i) = l_1, \Sigma_n^{-1}(j) = l_1\}$ occurs with $l_1 \in [n] - \{i, j\}$, then from the construction we have $\Sigma_n^{-1}(i) = j$. Using this with (3.9), it follows that conditioned on $\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\}$, the probability of $C_{n,i,j}$ is $\frac{i-1}{n-2} \frac{1}{n-k-1} + \frac{n-j}{n-2} \frac{1}{n-k-1} + \frac{1}{2} \frac{n-k-3}{n-k-1}$, which we write as $\frac{1}{(n-2)(n-k-1)} \left(i - 1 + n - j + \frac{1}{2}(n-2)(n-k-3) \right)$. Thus, we conclude that

(3.10)
$$P\left(\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\} \cap C_{n,i,j} \right) =$$

$$\frac{d_{n-1,k}}{(n-1)!} \frac{n-k-1}{n-1} \frac{n-k-1}{n} \frac{1}{(n-2)(n-k-1)} \left(i - 1 + n - j + \frac{1}{2}(n-2)(n-k-3) \right).$$

From (3.7), (3.8) and (3.10) we obtain

(3.11)
$$P(\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}) = \frac{d_{n-1,k}}{(n-1)!} \times$$

$$\left(\frac{k(n-k-1)(n-i-1)}{n(n-1)(n-2)} + \frac{(n-k-1)}{n(n-1)(n-2)} \left(i - 1 + n - j + \frac{1}{2}(n-2)(n-3-k) \right) \right).$$

We now turn to the calculation of $P\left(\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap C_{n,i,j} \right)$. Similar to before, we write

(3.12)
$$\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} =$$
$$\left(\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\} \right) \cup \left(\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\} \right).$$

We have $P(\mathcal{D}_{n-1,k+1}) = \frac{d_{n-1,k+1}}{(n-1)!}$. Given $\mathcal{D}_{n-1,k+1}$, the probability that $\Sigma_{n-1}^{-1}(i) = i$ is $\frac{k+1}{n-1}$. Given $\mathcal{D}_{n-1,k+1}$, the event $\mathcal{D}_{n,k}$ will occur if and only if person j entering at stage n sits at a table that has exactly one person already. (This reduces the number of fixed points from $k+1$ at stage $n-1$ to k at stage n.) The probability of this is $\frac{k+1}{n}$. We now need to calculate the probability of $C_{n,i,j}$, given $\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\}$. Given this event, person j has probability $\frac{1}{k+1}$ of joining i in which case the event $C_{n,i,j}$ does not occur. With probability $\frac{k}{k+1}$, person j will join a table with a person other than i, and this person is uniformly distributed

over $[n] - \{i, j\}$. Thus given the above event, the probability of $C_{n,i,j}$ is $\frac{k}{k+1} \frac{n-1-i}{n-2}$.
Thus, we conclude that

(3.13)
$$P\left(\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) = i\} \cap C_{n,i,j}\right) =$$
$$\frac{d_{n-1,k+1}}{(n-1)!} \frac{k+1}{n-1} \frac{k+1}{n} \frac{k}{k+1} \frac{n-i-1}{n-2}.$$

We now calculate $P\left(\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\} \cap C_{n,i,j}\right)$. Given $\mathcal{D}_{n-1,k+1}$,
the probability that $\Sigma_{n-1}^{-1}(i) \neq i$ is $\frac{n-2-k}{n-1}$. As noted above, given $\mathcal{D}_{n-1,k+1}$, the
event $\mathcal{D}_{n,k}$ will occur if and only if person j entering at stage n sits at a table
that has exactly one person already, and the probability of this is $\frac{k+1}{n}$. We now
need to calculate the probability of $C_{n,i,j}$, given $\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\}$.
Given this event, $\Sigma_n^{-1}(i)$ and $\Sigma_n^{-1}(j)$ are independent and uniformly distributed
over $[n] - \{i, j\}$; thus the probability of $C_{n,i,j}$ is $\frac{1}{2}$. Thus, we conclude that

(3.14)
$$P\left(\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap \{\Sigma_{n-1}^{-1}(i) \neq i\} \cap C_{n,i,j}\right) =$$
$$\frac{d_{n-1,k+1}}{(n-1)!} \frac{n-2-k}{n-1} \frac{k+1}{n} \frac{1}{2}.$$

From (3.12)–(3.14), we obtain
(3.15)
$$P(\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}) = \frac{d_{n-1,k+1}}{(n-1)!} \left(\frac{(k+1)k(n-i-1)}{n(n-1)(n-2)} + \frac{(n-2-k)(k+1)}{2n(n-1)} \right).$$

From (3.2), (3.6), (3.11) and (3.15), it follows that $P_n(D_{n,k} \cap \{\sigma_n^{-1}(i) < \sigma_n^{-1}(j)\}) = P(\mathcal{D}_{n,k} \cap C_{n,i,j})$ is equal to the sum of the terms on the right hand sides of (3.6),
(3.11) and (3.15). Of course, $P_n^{(k)}(\sigma_n^{-1}(i) < \sigma_n^{-1}(j)) = \frac{1}{P_n(D_{n,k})} P_n(D_{n,k} \cap \{\sigma_n^{-1}(i) < \sigma_n^{-1}(j)\})$. Also

(3.16)
$$E_n^{(k)} I_n = \frac{n(n-1)}{2} - \sum_{1 \leq i < j \leq n} P_n^{(k)}(\sigma_i^{-1} < \sigma_j^{-1}).$$

Thus to calculate $E_n^{(k)} I_n$, we will calculate

$$\sum_{1 \leq i < j \leq k} P_n(D_{n,k} \cap \{\sigma_n^{-1}(i) < \sigma_n^{-1}(j)\}),$$

which is the sum of (3.6), (3.11) and (3.15) over the pairs i, j satisfying $1 \leq i < j \leq n$.

Using (2.21), we have for the sum of (3.6) over the pairs i, j satisfying $1 \leq i < j \leq n$,

(3.17)
$$\sum_{1 \leq i < j \leq n} P(\mathcal{D}_{n-1,k-1} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}) =$$

$$\sum_{1 \leq i < j \leq n} \frac{d_{n-1,k-1}}{(n-1)!} \left(\frac{k-1}{n(n-1)} + \frac{(n-k)(j-2)}{n(n-1)(n-2)} \right) = \frac{d_{n-1,k-1}}{(n-1)!} \times$$

$$\left(\frac{k-1}{n(n-1)} \frac{n(n-1)}{2} - \frac{2(n-k)}{n(n-1)(n-2)} \frac{n(n-1)}{2} + \frac{n-k}{n(n-1)(n-2)} \frac{(n-1)n(n+1)}{3} \right) =$$

$$\frac{d_{n-1,k-1}}{(n-1)!} \left(\frac{k-1}{2} + \frac{n-k}{3} \right).$$

We now consider the sum of (3.11) over the pairs i, j satisfying $1 \leq i < j \leq n$. Using (2.21), the sum of the first term on the second line of (3.11) is

$$\sum_{1 \leq i < j \leq n} \frac{k(n-k-1)(n-i-1)}{n(n-1)(n-2)} =$$

(3.18)
$$\frac{k(n-k-1)(n-1)}{n(n-1)(n-2)} \frac{n(n-1)}{2} - \frac{k(n-k-1)}{n(n-1)(n-2)} \frac{(n-1)n(n+1)}{6} =$$

$$k \frac{n-k-1}{n-2} \left(\frac{n-1}{2} - \frac{n+1}{6} \right).$$

Using (2.21), the sum of the second term on the second line of (3.11) is

(3.19)
$$\sum_{1 \leq i < j \leq n} \frac{(n-k-1)}{n(n-1)(n-2)} \left(i - 1 + n - j + \frac{1}{2}(n-2)(n-3-k) \right) =$$

$$\frac{n-k-1}{n(n-1)(n-2)} \left(n - 1 + \frac{1}{2}(n-2)(n-3-k) \right) \frac{n(n-1)}{2} -$$

$$\frac{n-k-1}{n(n-1)(n-2)} \frac{(n-1)n(n+1)}{6} = \frac{n-k-1}{n-2} \left(\frac{n-1}{2} + \frac{(n-2)(n-3-k)}{4} - \frac{n+1}{6} \right).$$

Thus, from (3.11), (3.18) and (3.19), we obtain

(3.20)
$$\sum_{1 \leq i < j \leq n} P(\mathcal{D}_{n-1,k} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}) = \frac{d_{n-1,k}}{(n-1)!} \frac{n-k-1}{n-2} \times$$

$$\left(\frac{k(n-1)}{2} - \frac{k(n+1)}{6} + \frac{n-1}{2} + \frac{(n-2)(n-3-k)}{4} - \frac{n+1}{6} \right) =$$

$$\frac{d_{n-1,k}}{(n-1)!} \frac{n-k-1}{12(n-2)} \left(3n^2 - 11n + (n-2)k + 10 \right).$$

Now we consider the sum of (3.15) over the pairs i, j satisfying $1 \leq i < j \leq n$. Using (2.21), we have

(3.21)
$$\sum_{1 \leq i < j \leq n} P(\mathcal{D}_{n-1,k+1} \cap \mathcal{D}_{n,k} \cap C_{n,i,j}) =$$

$$\sum_{1 \leq i < j \leq n} \frac{d_{n-1,k+1}}{(n-1)!} \left(\frac{(k+1)k(n-i-1)}{n(n-1)(n-2)} + \frac{(n-2-k)(k+1)}{2n(n-1)} \right) =$$

$$\frac{d_{n-1,k+1}}{(n-1)!} \left(\left(\frac{(k+1)k(n-1)}{n(n-1)(n-2)} + \frac{(n-2-k)(k+1)}{2n(n-1)} \right) \frac{n(n-1)}{2} - \right.$$

$$\left. \frac{(k+1)k}{n(n-1)(n-2)} \frac{(n-1)n(n+1)}{6} \right) = \frac{d_{n-1,k+1}}{(n-1)!} \left(\frac{k+1}{4} n + \frac{1}{12}(k+1)(k-6) \right).$$

We conclude that $\sum_{1 \leq i < j \leq n} P_n(D_n \cap \{\sigma_n^{-1}(i) < \sigma_n^{-1}(j)\})$, is the sum of the right hand sides of (3.17), (3.20) and (3.21). Consequently, $\sum_{1 \leq i < j \leq n} P_n^{(k)}(\sigma_n^{-1}(i) < \sigma_n^{-1}(j))$ is the above noted sum divided by $P_n(D_{n,k})$. It is easy to see that $d_{m,r} = \binom{m}{r} d_{m-r}$, for $0 \leq r \leq m$. Thus,

(3.22)
$$\frac{d_{m,r}}{m!} = \frac{1}{r!} \frac{d_{m-r}}{(m-r)!} = \frac{1}{r!} \sum_{l=0}^{m-r} \frac{(-1)^l}{l!},$$

where the last equality follows from (2.6). For convenience in notation, let

(3.23)
$$\mathcal{E}_m(-1) = \sum_{l=0}^{m} \frac{(-1)^l}{l!}.$$

Thus, from (3.22),

$$P_n(D_{n,k}) = \frac{d_{n,k}}{n!} = \frac{1}{k!} \sum_{l=0}^{n-k} \frac{(-1)^l}{l!} = \frac{1}{k!} \mathcal{E}_{n-k}(-1).$$

Using this with (3.22), we have

(3.24)
$$\frac{\frac{d_{n-1,k-1}}{(n-1)!}}{P_n(D_{n,k})} = k;$$

$$\frac{\frac{d_{n-1,k}}{(n-1)!}}{P_n(D_{n,k})} = 1 - \frac{\frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)};$$

$$\frac{\frac{d_{n-1,k+1}}{(n-1)!}}{P_n(D_{n,k})} = \frac{1}{k+1} \left(1 - \frac{\frac{(-1)^{n-k-1}}{(n-k-1)!} + \frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)} \right).$$

From the facts noted in the previous paragraph along with (3.24), (3.17), (3.20) and (3.21), we obtain

$$\sum_{1 \le i < j \le n} P_n^{(k)}(\sigma_n^{-1}(i) < \sigma_n^{-1}(j)) = k\left(\frac{k-1}{2} + \frac{n-k}{3}\right) +$$

$$\left(1 - \frac{\frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)}\right)\left(\frac{n-k-1}{12(n-2)}\left(3n^2 - 11n + (n-2)k + 10\right)\right) +$$

$$\frac{1}{k+1}\left(1 - \frac{\frac{(-1)^{n-k-1}}{(n-k-1)!} + \frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)}\right)\left(\frac{k+1}{4}n + \frac{1}{12}(k+1)(k-6)\right).$$

From this and (3.16), we obtain

$$E_n^{(k)} I_n = \frac{n(n-1)}{2} - k\left(\frac{k-1}{2} + \frac{n-k}{3}\right) -$$

(3.25)
$$\left(1 - \frac{\frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)}\right)\left(\frac{n-k-1}{12(n-2)}\left(3n^2 - 11n + (n-2)k + 10\right)\right) -$$

$$\left(1 - \frac{\frac{(-1)^{n-k-1}}{(n-k-1)!} + \frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)}\right)\left(\frac{n}{4} + \frac{1}{12}(k-6)\right).$$

Now

(3.26) $\quad \dfrac{n-k-1}{12(n-2)}(3n^2 - 11n + (n-2)k + 10) = \dfrac{1}{4}n^2 - \dfrac{4+k}{6}n - \dfrac{1}{12}(k+1)(k-5).$

Consequently, the summands on the right hand side of (3.25) that do not involve $\mathcal{E}_n(-1)$ are given by

(3.27)
$$\frac{n(n-1)}{2} - k\left(\frac{k-1}{2} + \frac{n-k}{3}\right) - \frac{1}{4}n^2 + \frac{4+k}{6}n + \frac{1}{12}(k+1)(k-5) - \frac{n}{4} - \frac{1}{12}(k-6) =$$

$$\frac{n(n-1)}{4} - \frac{k-1}{6}n - \frac{k^2 - k - 1}{12}.$$

Writing $\frac{(-1)^{n-k-1}}{(n-k-1)!} = -(n-k)\frac{(-1)^{n-k}}{(n-k)!}$ and using (3.26), we can write the summands on the right hand side of (3.25) that involve $\mathcal{E}_n(-1)$ as

(3.28)
$$\frac{\frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)}\left(\frac{1}{4}n^2 - \frac{4+k}{6}n - \frac{1}{12}(k+1)(k-5) + \left(-(n-k) + 1\right)\left(\frac{n}{4} + \frac{1}{12}(k-6)\right)\right) =$$

$$\frac{\frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)}\left(\frac{n-k-1}{12}\right).$$

Now (1.7) follows from (3.25), (3.27), (3.28) and (3.23).

We now prove (1.8). From the first two sentences in the paragraph following (3.15), along with (3.24), it follows that

(3.29)
$$P_n^{(k)}(\sigma_n^{-1}(i) < \sigma_n^{-1}(j)) = k\left(\frac{k-1}{n(n-1)} + \frac{(n-k)(j-2)}{n(n-1)(n-2)}\right) +$$

$$\left(1 - \frac{\frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)}\right) \times$$

$$\left(\frac{k(n-k-1)(n-i-1)}{n(n-1)(n-2)} + \frac{(n-k-1)}{n(n-1)(n-2)}\left(i-1+n-j+\frac{1}{2}(n-2)(n-3-k)\right)\right) +$$

$$\frac{1}{k+1}\left(1 - \frac{\frac{(-1)^{n-k-1}}{(n-k-1)!} + \frac{(-1)^{n-k}}{(n-k)!}}{\mathcal{E}_{n-k}(-1)}\right) \times$$

$$\left(\frac{(k+1)k(n-i-1)}{n(n-1)(n-2)} + \frac{(n-2-k)(k+1)}{2n(n-1)}\right).$$

With a lot of algebra, one can show that the summands on the right hand side of (3.29) that do not involve $\mathcal{E}_n(-1)$ satisfy

(3.30)
$$k\left(\frac{k-1}{n(n-1)} + \frac{(n-k)(j-2)}{n(n-1)(n-2)}\right) +$$

$$\frac{k(n-k-1)(n-i-1)}{n(n-1)(n-2)} + \frac{(n-k-1)}{n(n-1)(n-2)}\left(i-1+n-j+\frac{1}{2}(n-2)(n-3-k)\right) +$$

$$\frac{k(n-i-1)}{n(n-1)(n-2)} + \frac{(n-2-k)}{2n(n-1)} =$$

$$\frac{1}{2} + \frac{1}{2n(n-1)(n-2)}\left((2(k-1)(j-i) + (k^2-3k+1)n - 2(k^2-k-1)(j-i)\right).$$

Now (1.8) follows from (3.29), (3.30) and (3.23)   □

With considerably more algebra, one can show that the summands on the right hand side of (3.29) that involve $\mathcal{E}_n(-1)$ can be written as

(3.31)
$$\frac{\frac{(-1)^{n-k}}{(n-k)!}}{\sum_{l=0}^{n-k} \frac{-1)^l}{l!}} \frac{1}{2n(n-1)(n-2)} \times$$

$$\left(kn^3 - (5k+1)n^2 + (-k^3(3-2i)k^2 + 9k + 2(j-i)+1)n +\right.$$

$$\left. 2ik^3 + 2(i-2)k^2 - 2(j-i+2)k - 2(j-i)\right).$$

From (3.29)–(3.31) and (3.23), it follows that the exact formula for $P_n^{(k)}(\sigma_i^{-1} < \sigma_j^{-1})$ is

(3.32)
$$P_n^{(k)}(\sigma_i^{-1} < \sigma_j^{-1}) =$$
$$\frac{1}{2} + \frac{1}{2n(n-1)(n-2)}\Big(\big(2(k-1)(j-i) + k^2 - 3k + 1\big)\,n - 2(k^2 - k - 1)(j-i)\Big) +$$
$$\frac{\frac{(-1)^{n-k}}{(n-k)!}}{\sum_{l=0}^{n-k}\frac{-1)^l}{l!}}\frac{1}{2n(n-1)(n-2)}\times$$
$$\Big(kn^3 - (5k+1)n^2 + (-k^3(3-2i)k^2 + 9k + 2(j-i) + 1)n +$$
$$2ik^3 + 2(i-2)k^2 - 2(j-i+2)k - 2(j-i)\Big).$$

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