MEAN AND VARIANCE OF THE LONGEST ALTERNATING SUBSEQUENCE IN A RANDOM SEPARABLE PERMUTATION

ROSS G. PINSKY

ABSTRACT. A permutation is *separable* if it can be obtained from the singleton permutation by iterating direct sums and skew sums. Equivalently, it is separable if and only it avoids the patterns 2413 and 3142. Under the uniform probability on separable permutations of $[n]$, let the random variable A_n denote the length of the longest alternating subsequence. Also, let $A_n^{+,-}$ denote the length of the longest alternating subsequence that begins with an ascent and ends with a descent, and define $A_n^{-,+}, A_n^{+,+}, A_n^{-,-}$ similarly. By symmetry, the first two and the last two of these latter four random variables are equi-distributed. We prove that the expected value of any of these five random variables behaves asymptotically as $(2 - \sqrt{2})n \approx 0.5858\, n$. We also obtain the more refined estimates that the expected value of $A_n^{+,-}$ and of $A_n^{-,+}$ is equal to $(2 - \sqrt{2})n - \frac{1}{4}(3 - 2\sqrt{2}) + o(1)$ and that the expected value of $A_n^{+,+}$ and of $A_n^{-,-}$ is equal to $(2 - \sqrt{2})n + \frac{3}{4}(3 - 2\sqrt{2}) + o(1)$. Finally, we show that the variance of any of the four random variables $A_n^{\pm,\pm}$ behaves asymptotically as $\frac{16 - 11\sqrt{2}}{2}n \approx 0.2218\, n$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let S_n denote the permutations of $[n] := \{1, \cdots, n\}$. Given $\sigma \in S_k$ and $\tau \in S_l$, the *direct sum* of σ and τ is the permutation in S_{k+l} given by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i), \ i = 1, \cdots, k; \\ \tau(i - k) + k, \ i = k + 1, \cdots k + l, \end{cases}$$

and the *skew sum* $\sigma \ominus \tau$ is the permutation in S_{k+l} given by

$$(\sigma \ominus \tau)(i) = \begin{cases} \sigma(i) + l, \ i = 1, \cdots, k; \\ \tau(i - k), \ i = k + 1, \cdots k + l. \end{cases}$$

A permutation is *indecomposable* if it cannot be represented as the direct sum of two nonempty permutations and is *skew indecomposable* if it cannot be represented as the skew sum of two nonempty permutations. A permutation is *separable* if it can be obtained from the singleton permutation by iterating direct sums and skew sums. Equivalently, a permutation is separable if it can be successively decomposed and skew decomposed until all of the indecomposable and skew indecomposable pieces of the permutation are singletons. For example, using one-line notation, consider the separable permutation 4352167. It can be decomposed into $43521 \oplus 12$. Then 43521 can be skew decomposed into $213 \ominus 21$ and 12 can be decomposed into $1 \oplus 1$. Now 213 can be decomposed into $21 \oplus 1$ and 21 can be skew decomposed into $1 \ominus 1$. Finally, again 21 can be skew decomposed into $1 \ominus 1$.

It is well-known [3] that a permutation is separable if and only it avoids the patterns 2413 and 3142. For more on pattern avoiding permutations, see for example [2]. The fact that separable permutations can be enumerated by a closed form generating function (see section 2) makes them rather tractable to analyze. The study of general pattern avoiding permutations goes back to Knuth's observation [7] that a permutation is so-called stack sortable if and only if it 231-avoiding. Similarly, the study of separable permutations goes back to [1] where it was shown that these are precisely the permutations which are sortable by so-called pop stacks. Separable permutations also arise in a variety of other applications, for example in bootstrap percolation [8] and in connection to polynomial interchanges where one studies the possible ways that the relative order of the values of a family of polynomials can be modified when crossing a common zero [6].

Let SEP(n) denote the set of separable permutations in S_n, and let P_n^{sep} and E_n^{sep} denote respectively the uniform probability measure on SEP(n) and the expectation with respect to that measure. In this paper we study the length of the longest alternating subsequence in a random separable permutation. An alternating subsequence of length k in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is a subsequence of the form $\sigma_{i_1} > \sigma_{i_2} < \sigma_{i_3} > \cdots \sigma_{i_k}$ or

$\sigma_{i_1} < \sigma_{i_2} > \sigma_{i_3} < \cdots \sigma_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq n$. Let $A_n = A_n(\sigma)$ denote the length of the longest alternating subsequence in a permutation $\sigma \in S_n$. Stanley [9] showed that for a uniformly random permutation in S_n, the expectation of A_n is asymptotic to $\frac{2}{3}n$ and the variance is asymptotic to $\frac{8}{45}n$. The length of the longest alternating subsequence in a random permutation avoiding a pattern of length three was studied in [4]. It was proven that the expectation of A_n in a random η-avoiding permutation in S_n is asymptotic to $\frac{1}{2}n$ and the variance is asymptotic to $\frac{1}{4}n$, for any pattern $\eta \in S_3$.

We will prove the following theorem.

Theorem 1.

$$(1.1) \qquad\qquad E_n^{\mathrm{sep}} A_n \sim (2 - \sqrt{2})n \approx 0.5858\, n.$$

Remark. Since a separable permutation is one that avoids the two patterns 2413 and 3142, and since these patterns are alternating, it is perhaps not surprising that the expected length of the longest alternating subsequence in a random separable permutation is shorter than that of a uniformly random permutation. Note though that it is longer than that of a random permutation avoiding any particular pattern of length three.

For the proof of Theorem 1 and also in order to state a result for the asymptotic behavior of the variance, we need to differentiate between four types of alternating sequences—begin with an ascent and end with an ascent; begin with an ascent and end with a descent; begin with a descent and end with an ascent; begin with a descent and end with a descent. An alternating subsequence of length k which begins with an ascent and ends with an ascent in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is a subsequence of the form $\sigma_{i_1} < \sigma_{i_2} > \cdots < \sigma_{i_k}$. The other three types are defined similarly.

Denote the length of the longest alternating subsequence of each of the four types of alternating subsequences by $A_n^{\pm,\pm}$. Since the difference of any two of the random variables $A_n^{\pm,\pm}$ is bounded by two, Theorem 1 holds with A_n replaced by any of the four random variables $A_n^{\pm,\pm}$. From symmetry considerations, it is clear that

$$(1.2) \qquad\qquad A_n^{+,+} \overset{\mathrm{dist}}{=} A_n^{-,-}; \quad A_n^{+,-} \overset{\mathrm{dist}}{=} A_n^{-,+}.$$

We will prove the following theorem which refines Theorem 1 and gives the asymptotic behavior of the variance.

Theorem 2. *i.*

$$(1.3) \qquad E_n^{\mathrm{sep}} A_n^{+,-} = E_n^{\mathrm{sep}} A_n^{-,+} = (2 - \sqrt{2})n - \frac{1}{4}(3 - 2\sqrt{2}) + O\left(\frac{1}{n}\right);$$

$$(1.4) \qquad \mathrm{Var}_n^{\mathrm{sep}}(A_n^{+,-}) = \mathrm{Var}_n^{\mathrm{sep}}(A_n^{-,+}) \sim \frac{16 - 11\sqrt{2}}{2}n \approx 0.2218\, n.$$

ii.

$$(1.5) \qquad E_n^{\mathrm{sep}} A_n^{+,+} = E_n^{\mathrm{sep}} A_n^{-,-} = (2 - \sqrt{2})n + \frac{3}{4}(3 - 2\sqrt{2}) + O\left(\frac{1}{n}\right);$$

$$(1.6) \qquad \mathrm{Var}_n^{\mathrm{sep}}(A_n^{+,+}) = \mathrm{Var}_n^{\mathrm{sep}}(A_n^{-,-}) \sim \frac{16 - 11\sqrt{2}}{2}n \approx 0.2218\, n.$$

Remark. From Theorem 2, it follows that from the limited perspective of mean and variance, $A_n^{+,+}$ or $A_n^{-,-}$ behaves as a deterministic translation of $A_n^{+,-}$ or $A_n^{-,+}$ by $3 - 2\sqrt{2}$. We note that the quantity $3 - 2\sqrt{2}$ plays a fundamental role in the proofs; it is one of the roots of the generating function corresponding to the count of separable permutations–see Section 2.

In Section 2 we present some preliminary material on separable permutations and on alternating subsequences. In Section 3, we define two generating functions related to the mean of the length of the longest alternating sequence and evaluate them explicitly. Using one of these generating functions, we prove Theorem 1 in Section 4. In Section 5, we define two generating functions related to the second moment of the length of the longest alternating sequence and evaluate them explicitly. Using all four of the above noted generating functions, we prove Theorem 2 in Section 6. In the appendix we present a proposition that provides the asymptotic analysis required in the proofs of Theorems 1 and 2.

2. Preliminary material

Let $s_n = |\mathrm{SEP}(n)|$, $n \geq 1$, denote the number of separable permutations in S_n. Let

$$s(t) = \sum_{n=1}^{\infty} s_n t^n$$

denote the generating function of $\{s_n\}_{n=1}^\infty$. For a separable permutation, define the length of the first indecomposable block and the length of the first skew indecomposable block respectively by

(2.1)
$$B_1^{+,n}(\sigma) = \min\{j : \sigma([j]) = [j]\}, \ \sigma \in \mathrm{SEP}(n);$$
$$B_1^{-,n}(\sigma) = \min\{k : \sigma([k]) = [n] \setminus [n-k]\}, \sigma \in \mathrm{SEP}(n).$$

By the definition of separable permutations, for each $\sigma \in \mathrm{SEP}(n)$, with $n \geq 2$, exactly one out of $B_1^{+,n}(\sigma)$ and $B_1^{-,n}(\sigma)$ is equal to n, and by symmetry,

(2.2)
$$|\{\sigma \in \mathrm{SEP}(n) : B_1^{+,n}(\sigma) = n\}| = |\{\sigma \in \mathrm{SEP}(n) : B_1^{-,n}(\sigma) = n\}| = \frac{1}{2}s_n, \ n \geq 2.$$

That is, half of the permutations in $\mathrm{SEP}(n)$, $n \geq 2$, are indecomposable and half are skew indecomposable. Partitioning $\mathrm{SEP}(n)$ by $\{B_1^{+,n} = j\}_{j=1}^n$ (or alternatively, by $\{B_1^{-,n} = j\}_{j=1}^n$), and using the concatenating structure of separable permutations, it follows that

(2.3)
$$s_n = s_1 s_{n-1} + \frac{1}{2}\sum_{j=2}^{n-1} s_j s_{n-j} + \frac{1}{2}s_n, \ n \geq 2,$$

and

(2.4)
$$P_n^{\mathrm{sep}}(B_1^{+,n} = j) = P_n^{\mathrm{sep}}(B_1^{-,n} = j) = \begin{cases} \frac{s_1 s_{n-1}}{s_n}, & j = 1; \\ \frac{\frac{1}{2}s_j s_{n-j}}{s_n}, & j = 2 \cdots n-1; \\ \frac{1}{2}, & j = n. \end{cases}$$

From (2.3) it is straightforward to show that

(2.5)
$$s(t) = \frac{1}{2}(1 - t - \sqrt{t^2 - 6t + 1}), \ \text{for } |t| < 3 - 2\sqrt{2}.$$

(Multiply both sides of (2.3) by t^n and sum over n from 2 to ∞, and then solve for $s(t)$.) Using the above formula for the generating function, one can prove that

$$s_n \sim \frac{1}{2^{\frac{3}{4}}\sqrt{\pi n^3}}(3 - 2\sqrt{2})^{-n+\frac{1}{2}}.$$

A more refined version of this asymptotic formula can be found in (7.2) in Proposition 1 in the appendix. We note that in [5, p. 474-475], the above formula appears with a mistake—instead of $2^{\frac{3}{4}}$, one finds there 2. (Our s_n is equal to their D_{n-1}.) The sequence of integers $\{s_n\}_{n=1}^\infty$ is known as the sequence of big Schröder numbers; see A006318 in the *On-Line Encyclopedia of Integer Sequences*.

As noted before the statement of Theorem 2, we differentiate between four types of alternating sequences, depending on whether they begin with an ascent or a descent and whether they end with an ascent or a descent, and denote the length of the longest one of each type by $A_n^{\pm,\pm}$. We note that a subsequence of length two of the form $\sigma_{i_1} < \sigma_{i_2}$ begins with an ascent and ends with an ascent. We will derive recursion formulas that allow us to obtain generating functions in explicit forms. In order to make the formulas work, the following definition will be crucial:

(2.6)
A singleton σ_{i_1} is considered an alternating sequence both of the type $(+,-)$ and of the type $(-,+)$;

A singleton σ_{i_1} is not considered an alternating sequence of the type $(+,+)$ or of the type $(-,-)$.

In light of (2.6), note that

$$A_1^{+,+} = A_1^{-,-} \equiv 0; \quad A_1^{+,-} = A_1^{-,+} \equiv 1;$$

$$A_2^{+,-} = A_2^{-,+} \equiv 1$$

(2.7)
$$A_2^{+,+}(\sigma) = \begin{cases} 2, & \sigma = 12, \\ 0, & \sigma = 21; \end{cases} \quad A_2^{-,-}(\sigma) = \begin{cases} 0, & \sigma = 12, \\ 2, & \sigma = 21. \end{cases}$$

We remark that $A_n^{+,+}$ and $A_n^{-,-}$ take on only even values, and $A_n^{+,-}$ and $A_n^{-,+}$ take on only odd values.

The following lemma will play an important role. Let $P_n^{\text{sep}}(\cdot \,|B_1^{+,n} = n)$ $\left(P_n^{\text{sep}}(\cdot \,|B_1^{-,n} = n)\right)$ denote the distribution P_n^{sep} conditioned on $\{B_1^{+,n} = n\}$ $\left(\{B_1^{-,n} = n\}\right)$.

Lemma 1. *i. The distribution of $A_n^{+,+}$ $\left(A_n^{-,-}\right)$ under $P_n^{\text{sep}}(\cdot \,|B_1^{+,n} = n)$ coincides with the distribution of $A_n^{-,-}$ $\left(A_n^{+,+}\right)$ under $P_n^{\text{sep}}(\cdot \,|B_1^{-,n} = n)$;*
ii. The distribution of $A_n^{+,-}$ $\left(A_n^{-,+}\right)$ under $P_n^{\text{sep}}(\cdot \,|B_1^{+,n} = n)$ coincides with the distribution of $A_n^{-,+}$ $\left(A_n^{+,-}\right)$ under $P_n^{\text{sep}}(\cdot \,|B_1^{-,n} = n)$;
iii. The distributions of $A_n^{+,-}$ and $A_n^{-,+}$ coincide under $P_n^{\text{sep}}(\cdot \,|B_1^{+,n} = n)$.

Proof. Recall that the *reverse* of a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is the permutation $\sigma^{\text{rev}} := \sigma_n \cdots \sigma_1$, and the *complement* of σ is the permutation σ^{com}

satisfying $\sigma_i^{\mathrm{com}} = n + 1 - \sigma_i$, $i = 1, \cdots, n$. Let $\sigma^{\mathrm{rev\text{-}com}}$ denote the permutation obtained by applying reversal and then complementation to σ (or equivalently, vice versa). All of these operations are bijections of $\mathrm{SEP}(n)$; indeed, they all preserve the property of being simultaneously 2413 and 3142 avoiding. It is easy to see that

$$B_1^{+,n}(\sigma) = n \; \Leftrightarrow \; B_1^{-,n}(\sigma^{\mathrm{rev}}) = n \; \Leftrightarrow \; B_1^{-,n}(\sigma^{\mathrm{com}}) = n, \; \sigma \in S_n.$$

From the above facts, it follows that

(2.8)
$\sigma \to \sigma^{\mathrm{com}}$ is a bijection of $\mathrm{SEP}(n) \cap \{B_1^{+,n} = n\}$ to $\mathrm{SEP}(n) \cap \{B_1^{-,n} = n\}$;

$\sigma \to \sigma^{\mathrm{rev\text{-}com}}$ is a bijection of $\mathrm{SEP}(n) \cap \{B_1^{+,n} = n\}$ to itself.

Also, it is easy to check that

$$A_n^{+,+}(\sigma) = A_n^{-,-}(\sigma^{\mathrm{com}}), \; \sigma \in S_n;$$

(2.9)
$$A_n^{+,-}(\sigma) = A_n^{-,+}(\sigma^{\mathrm{com}});$$

$$A_n^{+,-}(\sigma) = A_n^{+,-}(\sigma^{\mathrm{rev}}).$$

From the latter two equations in (2.9), it follows that

(2.10)
$$A_n^{+,-}(\sigma) = A_n^{-,+}(\sigma^{\mathrm{rev\text{-}com}}).$$

Part (i) of the lemma follows from the first line of (2.8) and the first line of (2.9); part (ii) follows from the first line of (2.8) and the second line of (2.9); and part (iii) follows from the second line of (2.8) and (2.10).   □

3. Generating functions related to the mean of A_n

Define

(3.1) $c_n^{\pm,\pm} = E_n^{\mathrm{sep}}(A_n^{\pm,\pm}|B_1^{+,n} = n); \quad d_n^{\pm,\pm} = E_n^{\mathrm{sep}}(A_n^{\pm,\pm}|B_1^{-,n} = n).$

Define the generating functions

$$G^{+,-}(t) = \sum_{n=1}^{\infty} s_n c_n^{+,-} t^n;$$

(3.2)

$$G^{-,-}(t) = \sum_{n=1}^{\infty} s_n c_n^{-,-} t^n.$$

We will prove the following theorem.

Theorem 3.

$$G^{+,-}(t) = \frac{t(1-t)}{\sqrt{t^2 - 6t + 1}};$$

(3.3)

$$G^{-,-}(t) = \frac{t(1-t)}{\sqrt{t^2 - 6t + 1}} - \frac{t(1-t)}{2} - \frac{t}{2}\sqrt{t^2 - 6t + 1}.$$

Proof. By (2.2) we have

(3.4) $$E_n^{\text{sep}} A_n^{\pm,\pm} = \frac{1}{2} c_n^{\pm,\pm} + \frac{1}{2} d_n^{\pm,\pm}.$$

Conditioning on $\{B_1^{+,n} = j\}, j = 1, \cdots, n$, we obtain the equation

(3.5) $$E_n^{\text{sep}} A_n^{\pm,\pm} = \sum_{j=1}^{n} P_n^{\text{sep}}(B_1^{+,n} = j) E_n^{\text{sep}}(A_n^{\pm,\pm} | B_1^{+,n} = j).$$

Consider (3.5) with $A_n^{+,+}$. From the definition of $A_n^{+,+}$ and the concatenating structure of separable permutations as manifested in (2.3), we have

(3.6)

$$A_n^{+,+}|\{B_1^{+,n} = j\} \stackrel{\text{dist}}{=} A_j^{+,-}|\{B_1^{+,j} = j\} + A_{n-j}^{-,+}, \; j = 1, \cdots, n-1, \; n \geq 3.$$

where the random variable on the left hand side is considered under P_n^{sep}, the random variable $A_j^{+,-}|\{B_1^{+,j} = j\}$ is considered under P_j^{sep}, the random variable $A_{n-j}^{-,+}$ is considered under P_{n-j}^{sep}, and $A_j^{+,-}|\{B_1^{+,j} = j\}$ and $A_{n-j}^{-,+}$ are independent.

To illustrate (3.6), we give three examples of what can occur. Consider first the permutation 342178956, which satisfies $B_1^{+,9}(342178956) = 4$. So $n = 9$, $j = 4$ and $n-j = 5$. The length of the longest alternating subsequence that begins and ends with an ascent is six. Such an alternating subsequence is built from a longest alternating subsequence beginning with an ascent and ending with a descent that appears in the first j entries of the permutation— 3421, and then concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n - j$ entries of the permutation— 78956. There are two possibilities for the first piece, namely 342 and 341, and there are three possibilities for the second piece, namely 756 and 856 and 956. Consider now the permutation 124378956, which satisfies $B_1^{+,9}(124378956) = 1$. So $n = 9$, $j = 1$ and $n - j = 8$. The length of the longest alternating subsequence that begins and ends with an ascent is six. Such an alternating subsequence is built from a longest alternating subsequence beginning with an ascent and ending with a descent

that appears in the first j entries of the permutation— 1, and concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n - j$ entries of the permutation— 24378956. There is one possibility for the first piece, namely 1, and there are three possibilities for the second piece, namely 43756, 43856 and 43956. Note that for this to work, it was necessary in (2.6) that a singleton be defined as an alternating sequence of the type $(+, -)$. Finally, consider the permutation 324561789, which satisfies $B_1^{+,9}(324561789) = 6$. So $n = 9$, $j = 6$ and $n - j = 3$. The length of the longest alternating subsequence that begins and ends with an ascent is four. Such an alternating subsequence is built from a longest alternating subsequence beginning with an ascent and ending with a descent that appears in the first j entries of the permutation— 324561, and then concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n - j$ entries of the permutation— 789. There are six possibilities for the first piece, namely 341,351,361,241,251,261, and there are three possibilities for the second piece, namely 7, 8, 9. Note that for this to work, it was necessary in (2.6) that a singleton be defined as an alternating sequence of the type $(-, +)$.

Now consider (3.5) with $A_n^{-,+}$. From the definition of $A_n^{-,+}$ and the concatenating structure of separable permutations as manifested in (2.3), we have

(3.7)
$$A_n^{-,+}|\{B_1^{+,n} = j\} \stackrel{\text{dist}}{=} A_j^{-,-}|\{B_1^{+,j} = j\} + A_{n-j}^{-,+}, \ j = 1, \cdots, n - 1, \ n \geq 3.$$

To illustrate (3.7), we give two examples of what can occur. Consider first the permutation 342178956, which satisfies $B_1^{+,n}(342178956) = 4$. So $n = 9, j = 4$ and $n - j = 5$. The length of the longest alternating subsequence that begins with a descent and ends with an ascent is five. Such an alternating subsequence is built from a longest alternating subsequence beginning with a descent and ending with a descent that appears in the first j entries of the permutation— 3421, and then concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n - j$ entries of the permutation— 78956. There are four possibilities for the first piece, namely 32, 42, 31, 41, and there are three possibilities for

the second piece, namely, 756, 856, 956. Consider now the permutation 145678923, which satisfies $B_1^{+,9}(145678923) = 1$. So $n = 9$, $j = 1$ and $n-j = 8$. The length of the longest alternating subsequence that begins with a descent and ends with an ascent is three. Such an alternating subsequence is built from a longest alternating subsequence beginning with a descent and ending with a descent that appears in the first j entries of the permutation— 1, and then concatenating this with a longest alternating subsequence that begins with a descent and ends with an ascent in the last $n-j$ entries of the permutation— 45678923. There is one possibility for the first piece, namely the null set, and there are six possibilities for the second piece, namely $x23$, with $x \in \{4,5,6,7,8,9\}$. Note that for this to work, it was necessary in (2.6) that a singleton be defined *not* to be an alternating sequence of the type $(-,-)$.

Taking expectations in (3.6), and using (3.5) along with (3.4) and (2.4), we obtain

$$E_n^{\mathrm{sep}} A_n^{+,+} = \frac{1}{2}c_n^{+,+} + \frac{1}{2}d_n^{+,+} = \frac{s_1 s_{n-1}}{s_n}\left(c_1^{+,-} + \frac{1}{2}c_{n-1}^{-,+} + \frac{1}{2}d_{n-1}^{-,+}\right) +$$

$$\sum_{j=2}^{n-1} \frac{\frac{1}{2}s_j s_{n-j}}{s_n}\left(c_j^{+,-} + \frac{1}{2}c_{n-j}^{-,+} + \frac{1}{2}d_{n-j}^{-,+}\right) + \frac{1}{2}c_n^{+,+}, n \geq 3,$$

or equivalently (noting from (2.7) that $c_1^{+,-} = 1$),
(3.8)
$$\frac{1}{2}d_n^{+,+} = \frac{s_1 s_{n-1}}{s_n}\left(1 + \frac{1}{2}c_{n-1}^{-,+} + \frac{1}{2}d_{n-1}^{-,+}\right) + \sum_{j=2}^{n-1} \frac{\frac{1}{2}s_j s_{n-j}}{s_n}\left(c_j^{+,-} + \frac{1}{2}c_{n-j}^{-,+} + \frac{1}{2}d_{n-j}^{-,+}\right), n \geq 3.$$

Taking expectations in (3.7), and using (3.5) along with (3.4) and (2.4), we obtain

$$E_n^{\mathrm{sep}} A_n^{-,+} = \frac{1}{2}c_n^{-,+} + \frac{1}{2}d_n^{-,+} = \frac{s_1 s_{n-1}}{s_n}\left(c_1^{-,-} + \frac{1}{2}c_{n-1}^{-,+} + \frac{1}{2}d_{n-1}^{-,+}\right) +$$

$$\sum_{j=2}^{n-1} \frac{\frac{1}{2}s_j s_{n-j}}{s_n}\left(c_j^{-,-} + \frac{1}{2}c_{n-j}^{-,+} + \frac{1}{2}d_{n-j}^{-,+}\right) + \frac{1}{2}c_n^{-,+}, n \geq 3,$$

or equivalently (noting from (2.7) that $c_1^{-,-} = 0$),
(3.9)
$$\frac{1}{2}d_n^{-,+} = \frac{s_1 s_{n-1}}{s_n}\left(\frac{1}{2}c_{n-1}^{-,+} + \frac{1}{2}d_{n-1}^{-,+}\right) + \sum_{j=2}^{n-1} \frac{\frac{1}{2}s_j s_{n-j}}{s_n}\left(c_j^{-,-} + \frac{1}{2}c_{n-j}^{-,+} + \frac{1}{2}d_{n-j}^{-,+}\right), n \geq 3.$$

By Lemma 1, we can substitute in equations (3.8) and (3.9) so that each of them is given only in terms of $c_{\cdot}^{-,-}$ and $c_{\cdot}^{+,-}$. Indeed, by Lemma 1, we have

$$(3.10) \qquad d_n^{+,+} = c_n^{-,-}, \ c_n^{-,+} = c_n^{+,-}, d_n^{-,+} = c_n^{+,-}.$$

Using (3.10) to substitute in (3.8) and (3.9) and recalling that $s_1 = 1$, we obtain the two equations

$$
\frac{1}{2}c_n^{-,-} = \frac{s_{n-1}}{s_n}\left(1 + c_{n-1}^{+,-}\right) + \sum_{j=2}^{n-1} \frac{\frac{1}{2}s_j s_{n-j}}{s_n}\left(c_j^{+,-} + c_{n-j}^{+,-}\right), n \geq 3;
$$

(3.11)

$$
\frac{1}{2}c_n^{+,-} = \frac{s_{n-1}}{s_n}c_{n-1}^{+,-} + \sum_{j=2}^{n-1} \frac{\frac{1}{2}s_j s_{n-j}}{s_n}\left(c_j^{-,-} + c_{n-j}^{+,-}\right), n \geq 3.
$$

Multiplying (3.11) by $2s_n t^n$ and summing over n gives

(3.12)

$$
\sum_{n=3}^{\infty} s_n c_n^{-,-} t^n = 2t \sum_{n=3}^{\infty} s_{n-1}\left(1 + c_{n-1}^{+,-}\right) t^{n-1} + \sum_{n=3}^{\infty}\left(\sum_{j=2}^{n-1} s_j s_{n-j}\left(c_j^{+,-} + c_{n-j}^{+,-}\right)\right) t^n;
$$

$$
\sum_{n=3}^{\infty} s_n c_n^{+,-} t^n = 2t \sum_{n=3}^{\infty} s_{n-1}c_{n-1}^{+,-} t^{n-1} + \sum_{n=3}^{\infty}\left(\sum_{j=2}^{n-1} s_j s_{n-j}\left(c_j^{-,-} + c_{n-j}^{+,-}\right)\right) t^n.
$$

We have

(3.13)

$$
\sum_{n=3}^{\infty}\left(\sum_{j=2}^{n-1} s_j s_{n-j} c_j^{+,-}\right) t^n = \sum_{n=3}^{\infty}\left(\sum_{j=1}^{n-1} s_j s_{n-j} c_j^{+,-}\right) t^n - t\sum_{n=3}^{\infty} s_{n-1} t^{n-1} =
$$

$$
\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} s_j s_{n-j} c_j^{+,-}\right) t^n - t\sum_{n=2}^{\infty} s_{n-1} t^{n-1} =
$$

$$
\left(\sum_{n=1}^{\infty} s_n c_n^{+,-} t^n\right)\left(\sum_{n=1}^{\infty} s_n t^n\right) - t\sum_{n=2}^{\infty} s_{n-1} t^{n-1},
$$

and similarly,

(3.14)

$$
\sum_{n=3}^{\infty}\left(\sum_{j=2}^{n-1} s_j s_{n-j} c_{n-j}^{+,-}\right) t^n = \sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} s_j s_{n-j} c_{n-j}^{+,-}\right) t^n - t\sum_{n=2}^{\infty} s_{n-1} c_{n-1}^{+,-} t^{n-1} =
$$

$$
\left(\sum_{n=1}^{\infty} s_n t^n\right)\left(\sum_{n=1}^{\infty} s_n c_n^{+,-} t^n\right) - t\sum_{n=2}^{\infty} s_{n-1} c_{n-1}^{+,-} t^{n-1}.
$$

Also, noting from (2.7) that $c_1^{-,-} = 0$, we have

(3.15)
$$\sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} c_j^{-,-} \right) t^n = \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} s_j s_{n-j} c_j^{-,-} \right) t^n = \left(\sum_{n=1}^{\infty} s_n c_n^{-,-} t^n \right) \left(\sum_{n=1}^{\infty} s_n t^n \right).$$

Since the conditional probability measure $P_2^{\text{sep}}(\cdot \,|B_1^{+,2} = 2)$ gives probability one to the permutation 21, it follows that $c_2^{-,-} = 2$. From (2.7), $c_2^{+,-} = 1$, $c_1^{+,-} = 1$ and $c_1^{-,-} = 0$. Using these facts along with the fact that $s_1 = 1$, $s_2 = 2$, it follows from (3.12)-(3.15) and (3.2) that

(3.16)
$$G^{-,-}(t) - 4t^2 = 2t(s(t) - t) + 2t(G^{+,-}(t) - t) + G^{+,-}(t)s(t) - ts(t)+$$
$$G^{+,-}(t)s(t) - tG^{+,-}(t)$$

and

(3.17)
$$G^{+,-}(t) - 2t^2 - t = 2t(G^{+,-}(t) - t) + G^{-,-}(t)s(t) + G^{+,-}(t)s(t) - tG^{+,-}(t).$$

The equation in (3.16) simplifies to

(3.18)
$$G^{-,-}(t) = (2s(t) + t) \, G^{+,-}(t) + ts(t).$$

Using (3.18) to substitute for $G^{-,-}(t)$ in (3.17), and performing some algebra, we obtain

(3.19)
$$G^{+,-}(t) = \frac{t(1 + s^2(t))}{1 - t - s(t) - 2s^2(t) - ts(t)}.$$

From (2.5), we have

(3.20)
$$s^2(t) = \frac{1}{2} \left(t^2 - 4t + 1 - (1 - t)\sqrt{t^2 - 6t + 1} \right).$$

Using (2.5) and (3.20), we obtain

(3.21) $$1 - t - s(t) - 2s^2(t) - ts(t) = -\frac{1}{2}(t^2 - 6t + 1) - \frac{1}{2}(t - 3)\sqrt{t^2 - 6t + 1}$$

and

(3.22)
$$s^2(t) + 1 = \frac{3}{2} - 2t + \frac{1}{2}t^2 - \frac{1 - t}{2}\sqrt{t^2 - 6t + 1}.$$

Substituting (3.21) and (3.22) in (3.19), and multiplying the numerator and denominator by -2 yields

(3.23)
$$G^{+,-}(t) = \frac{t \left(-3 + 4t - t^2 + (1 - t)\sqrt{t^2 - 6t + 1} \right)}{t^2 - 6t + 1 + (t - 3)\sqrt{t^2 - 6t + 1}}.$$

Writing $-3 + 4t - t^2 = 1 - (t-2)^2 = (1 - t + 2)(1 + t - 2) = (t-3)(1-t)$, we have

(3.24)
$$\frac{t\left(-3 + 4t - t^2 + (1-t)\sqrt{t^2 - 6t + 1}\right)}{t^2 - 6t + 1 + (t-3)\sqrt{t^2 - 6t + 1}} =$$
$$t(1-t)\frac{t - 3 + \sqrt{t^2 - 6t + 1}}{t^2 - 6t + 1 + (t-3)\sqrt{t^2 - 6t + 1}} = \frac{t(1-t)}{\sqrt{t^2 - 6t + 1}}.$$

The formula for $G^{+,-}$ in (3.3) follows from (3.23) and (3.24). The formula for $G^{-,-}$ in (3.3) follows from the formula in (3.3) for $G^{+,-}$, (3.18) and (2.5), along with a little algebra. $\qquad\square$

4. Proof of Theorem 1

To prove the theorem, it suffices to prove (1.1) with $E_n^{\text{sep}} A_n^{+,-}$ in place of $E_n^{\text{sep}} A_n$, since $A_n(\sigma) - A_n^{+,-}(\sigma) \in \{0, 1, 2\}$, for all $\sigma \in S_n$. By Lemma 1, $d^{+,-} = c^{-,+} = c^{+,-}$. From this and (3.4), we have $E_n^{\text{sep}} A_n^{+,-} = c_n^{+,-}$. Thus, from (3.2), it follows that the coefficient of t^n in the power series for $G^{+,-}(t)$ is $s_n E_n^{\text{sep}} A_n^{+,-}$. By Theorem 3 along with Proposition 1, which appears in the appendix, the coefficient of t^n in $G^{+,-}(t)$ is $a_{n-1} - a_{n-2}$, where a_n satisfies (7.3). Thus, we have

(4.1)
$$s_n E_n^{\text{sep}} A_n^{+,-} = a_{n-1} - a_{n-2} \sim \frac{1}{2^{\frac{5}{4}}\sqrt{\pi} n^{\frac{1}{2}}}\left((3 - 2\sqrt{2})^{-n+\frac{1}{2}} - (3 - 2\sqrt{2})^{-n+\frac{3}{2}}\right) =$$
$$\frac{1}{2^{\frac{5}{4}}\sqrt{\pi} n^{\frac{1}{2}}}(3 - 2\sqrt{2})^{-n+\frac{1}{2}}(2\sqrt{2} - 2) = \frac{1}{2^{\frac{3}{4}}\sqrt{\pi} n^{\frac{1}{2}}}(3 - 2\sqrt{2})^{-n+\frac{1}{2}}(2 - \sqrt{2}).$$

By (2.5) and by (7.2) in Proposition 1 in the appendix, we have

$$s_n \sim \frac{1}{2^{\frac{3}{4}}\sqrt{\pi} n^{\frac{3}{2}}}(3 - 2\sqrt{2})^{-n+\frac{1}{2}}.$$

Using this with (4.1), we conclude that $E_n^{\text{sep}} A_n^{+,-} \sim (2 - \sqrt{2})n$, which completes the proof of the theorem. $\qquad\square$

5. Generating functions related to the second moment of A_n

Define

(5.1)
$$C_n^{\pm,\pm} = E_n^{\text{sep}}\left((A_n^{\pm,\pm})^2 | B_1^{+,n} = n\right); \quad D_n^{\pm,\pm} = E_n^{\text{sep}}\left((A_n^{\pm,\pm})^2 | B_1^{-,n} = n\right).$$

Define the generating functions

$$H^{+,-}(t) = \sum_{n=1}^{\infty} s_n C_n^{+,-} t^n;$$
$$H^{-,-}(t) = \sum_{n=1}^{\infty} s_n C_n^{-,-} t^n.$$

We will prove the following theorem.

Theorem 4.

(5.3)
$$H^{+,-}(t) = 2t^2(1-t)^2 \left(t^2 - 6t + 1\right)^{-\frac{3}{2}} + \frac{-5t^3 + 8t^2 - 3t}{t-3} \left(t^2 - 6t + 1\right)^{-\frac{1}{2}} +$$

$$\frac{1}{2}t^2(1-t)\left(t^2 - 6t + 1\right)^{\frac{1}{2}} - \frac{1}{2}t^2(1-t);$$

$$H^{-,-}(t) = H^{+,-}(t) - \frac{-5t^3 + 8t^2 - 3t}{t-3} - \frac{1}{2}t^2(1-t)(t^2 - 6t + 1)+$$

$$\frac{1}{2}t^2(1-t)(t^2 - 6t + 1)^{\frac{1}{2}} + \frac{1}{2}t(1-t) - \frac{1}{2}t(t^2 - 6t + 1)^{\frac{1}{2}} + 2t^2(1-t)(t^2 - 6t + 1)^{-\frac{1}{2}}.$$

Proof. The notation $E_n^{\text{sep}} \left((A_j^{\delta_1,\delta_2} + A_{n-j}^{\delta_3,\delta_4})^2 | B_1^{\delta_5,j} = j, B_1^{\delta_6,n-j} = n - j\right)$ is employed in several equations below, where $\delta_i \in \{+, -\}$, for $i = 1, \cdots, 6$. This indicates that $A_j^{\delta_1,\delta_2}$ and $A_{n-j}^{\delta_3,\delta_4}$ are independent, with $A_j^{\delta_1,\delta_2}$ considered under the measure $E_j^{\text{sep}}(\cdot \ | B_1^{\delta_5,j} = j)$ and $A_{n-j}^{\delta_3,\delta_4}$ considered under the measure $E_{n-j}^{\text{sep}}(\cdot \ | B_1^{\delta_6,n-j} = n - j)$.

We proceed in the manner of the proof of Theorem 1. Using (3.6), (2.4), (2.2) and (5.1), and noting from (2.7) that $A_1^{+,-} = 1$, we have

$$E_n^{\text{sep}}(A_n^{+,+})^2 = \frac{1}{2}C_n^{+,+} + \frac{1}{2}D_n^{+,+} =$$

$$\frac{s_1 s_{n-1}}{s_n} \left(\frac{1}{2}E\left((1 + A_{n-1}^{-,+})^2 | B_1^{+,n-1} = n - 1\right) + \frac{1}{2}E\left((1 + A_{n-1}^{-,+})^2 | B_1^{-,n-1} = n - 1\right)\right) +$$

$$\sum_{j=2}^{n-1} \frac{1}{2}\frac{s_j s_{n-j}}{s_n}\frac{1}{2}E\left((A_j^{+,-} + A_{n-j}^{-,+})^2 | B_1^{+,j} = j, B_1^{+,n-j} = n - j\right) +$$

$$\sum_{j=2}^{n-1} \frac{1}{2}\frac{s_j s_{n-j}}{s_n}\frac{1}{2}E\left((A_j^{+,-} + A_{n-j}^{-,+})^2 | B_1^{+,j} = j, B_1^{-,n-j} = n - j\right) + \frac{1}{2}C_n^{+,+}, \ n \geq 3.$$

Expanding the terms in the equation above, cancelling the term $\frac{1}{2}C_n^{+,+}$ from both sides and using the notation from (5.1) and (3.1), we have

$$\frac{1}{2}D_n^{+,+} = \frac{s_1 s_{n-1}}{s_n}\left(1 + c_{n-1}^{-,+} + \frac{1}{2}C_{n-1}^{-,+} + d_{n-1}^{-,+} + \frac{1}{2}D_{n-1}^{-,+}\right) +$$

$$\frac{1}{4}\sum_{j=2}^{n-1}\frac{s_j s_{n-j}}{s_n}\left(C_j^{+,-} + 2c_j^{+,-}c_{n-j}^{-,+} + C_{n-j}^{-,+}\right) + \frac{1}{4}\sum_{j=2}^{n-1}\frac{s_j s_{n-j}}{s_n}\left(C_j^{+,-} + 2c_j^{+,-}d_{n-j}^{-,+} + D_{n-j}^{-,+}\right),$$

$n \geq 3$.

By Lemma 1, $D_n^{+,+} = C_n^{-,-}, D_n^{-,+} = C_n^{+,-}, C_n^{-,+} = C_n^{+,-}, c_n^{-,+} = c_n^{+,-}, d_n^{-,+} = c_n^{+,-}$, for all n. Making these substitutions in the equation above, multiplying both sides by $2s_n$ and recalling that $s_1 = 1$ gives

(5.4)

$$s_n C_n^{-,-} = s_{n-1}\left(2 + 4c_{n-1}^{+,-} + 2C_{n-1}^{+,-}\right) + \sum_{j=2}^{n-1} s_j s_{n-j}\left(C_j^{+,-} + 2c_j^{+,-}c_{n-j}^{+,-} + C_{n-j}^{+,-}\right), \ n \geq 3.$$

To derive a second recursion equation similar to the one in (5.4) we consider $A_n^{-,+}$. Using (3.7), (2.4), (2.2) and (5.1) and noting from (2.7) that $A_1^{-,-} = 0$, we have

$$E_n^{\text{sep}}(A_n^{-,+})^2 = \frac{1}{2}C_n^{-,+} + \frac{1}{2}D_n^{-,+} =$$

$$\frac{s_1 s_{n-1}}{s_n}\left(\frac{1}{2}E\left((A_{n-1}^{-,+})^2|B_1^{+,n-1} = n-1\right) + \frac{1}{2}E\left((A_{n-1}^{-,+})^2|B_1^{-,n-1} = n-1\right)\right) +$$

$$\sum_{j=2}^{n-1}\frac{1}{2}\frac{s_j s_{n-j}}{s_n}\frac{1}{2}E\left((A_j^{-,-} + A_{n-j}^{-,+})^2|B_1^{+,j} = j, B_1^{+,n-j} = n-j\right) +$$

$$\sum_{j=2}^{n-1}\frac{1}{2}\frac{s_j s_{n-j}}{s_n}\frac{1}{2}E\left((A_j^{-,-} + A_{n-j}^{-,+})^2|B_1^{+,j} = j, B_1^{-,n-j} = n-j\right) + \frac{1}{2}C_n^{-,+}, \ n \geq 3.$$

Expanding the terms in the equation above, cancelling the term $\frac{1}{2}C_n^{-,+}$ from both sides and using the notation from (5.1) and (3.1), we have

$$\frac{1}{2}D_n^{-,+} = \frac{s_1 s_{n-1}}{s_n}\left(\frac{1}{2}C_{n-1}^{-,+} + \frac{1}{2}D_{n-1}^{-,+}\right) + \frac{1}{4}\sum_{j=2}^{n-1}\frac{s_j s_{n-j}}{s_n}\left(C_j^{-,-} + 2c_j^{-,-}c_{n-j}^{-,+} + C_{n-j}^{-,+}\right) +$$

$$\frac{1}{4}\sum_{j=2}^{n-1}\frac{s_j s_{n-j}}{s_n}\left(C_j^{-,-} + 2c_j^{-,-}d_{n-j}^{-,+} + D_{n-j}^{-,+}\right), \ n \geq 3.$$

By Lemma 1, $D_n^{-,+} = C_n^{+,-}, C_n^{-,+} = C_n^{+,-}, c_n^{-,+} = c_n^{+,-}, d_n^{-,+} = c_n^{+,-}$, for all n. Making these substitutions in the equation above, multiplying both sides

by $2s_n$ and recalling that $s_1 = 1$ gives

(5.5)
$$s_n C_n^{+,-} = 2s_{n-1} C_{n-1}^{+,-} + \sum_{j=2}^{n-1} s_j s_{n-j} \left(C_j^{-,-} + 2c_j^{-,-} c_{n-j}^{+,-} + C_{n-j}^{+,-} \right), \ n \geq 3.$$

Multiplying (5.4) and (5.5) by t^n and summing over n gives

(5.6)
$$\sum_{n=3}^{\infty} s_n C_n^{-,-} t^n = 2t \sum_{n=3}^{\infty} s_{n-1} t^{n-1} + 4t \sum_{n=3}^{\infty} s_{n-1} c_{n-1}^{+,-} t^{n-1} + 2t \sum_{n=3}^{\infty} s_{n-1} C_{n-1}^{+,-} t^{n-1} +$$

$$\sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} C_j^{+,-} \right) t^n + \sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} C_{n-j}^{+,-} \right) t^n + 2 \sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} c_j^{+,-} c_{n-j}^{+,-} \right) t^n$$

and

(5.7)
$$\sum_{n=3}^{\infty} s_n C_n^{+,-} t^n = 2t \sum_{n=3}^{\infty} s_{n-1} C_{n-1}^{+,-} t^{n-1} + \sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} C_j^{-,-} \right) t^n +$$

$$\sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} C_{n-j}^{+,-} \right) t^n + 2 \sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} c_j^{-,-} c_{n-j}^{+,-} \right) t^n.$$

We have $s_2 = 2$. By (2.7), $C_1^{+,-} = 1$ and $C_1^{-,-} = 0$. By the explanation in the paragraph after (3.15), we have $C_2^{-,-} = 4$ and $C_2^{+,-} = 1$. The first and second double sums on the second line of (5.6) satisfy (3.13) and (3.14) respectively with $c^{+,-}$ replaced by $C^{+,-}$. The third double sum on the second line of (5.6) satisfies similarly,

$$\sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} c_j^{+,-} c_{n-j}^{+,-} \right) t^n =$$

$$\left(\sum_{n=1}^{\infty} s_n c_n^{+,-} t^n \right)^2 - t \sum_{n=2}^{\infty} s_{n-1} c_{n-1}^{+,-} t^{n-1}.$$

The double sum on the first line of (5.7) satisfies (3.15) with with $c^{-,-}$ replaced by $C^{-,-}$ and the first double sum on the second line of (5.7) satisfies (3.14) with $c^{+,-}$ replaced by $C^{+,-}$. The second double sum on the second line of (5.7) satisfies similarly

$$\sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} s_j s_{n-j} c_j^{-,-} c_{n-j}^{+,-} \right) t^n = \left(\sum_{n=1}^{\infty} s_n c_n^{-,-} t^n \right) \left(\sum_{n=1}^{\infty} s_n c_n^{+,-} t^n \right).$$

Using the above facts with (5.2), we obtain from (5.6) and (5.7)

(5.8)
$$H^{-,-}(t) - 8t^2 = 2t(s(t) - t) + 4t\left(G^{+,-}(t) - t\right) + 2t\left(H^{+,-}(t) - t\right) +$$
$$s(t)H^{+,-}(t) - ts(t) + s(t)H^{+,-}(t) - tH^{+,-}(t) + 2(G^{+,-}(t))^2 - 2tG^{+,-}(t)$$

and

(5.9)
$$H^{+,-}(t) - 2t^2 - t = 2t\left(H^{+,-}(t) - t\right) + s(t)H^{-,-}(t) +$$
$$s(t)H^{+,-}(t) - tH^{+,-}(t) + 2G^{-,-}(t)G^{+,-}(t).$$

The equation in (5.8) simplifies to

(5.10) $$H^{-,-}(t) = \left(2s(t) + t\right)H^{+,-}(t) + ts(t) + 2tG^{+,-}(t) + 2(G^{+,-}(t))^2.$$

Using (5.10) to substitute for $H^{-,-}(t)$ in (5.9), and then solving for $H^{+,-}(t)$, one obtains after some algebra,

(5.11)
$$H^{+,-}(t) = \frac{t + ts^2(t) + 2G^{-,-}(t)G^{+,-}(t) + 2ts(t)G^{+,-}(t) + 2s(t)(G^{+,-}(t))^2}{1 - t - s(t) - 2s^2(t) - ts(t)}.$$

Substituting for $G^{+,-}(t)$ and $G^{-,-}(t)$ from (3.3), substituting for $s(t)$ from (2.5) and substituting for $s^2(t)$ from (3.20), and performing a lot of algebra, one finds that the numerator in (5.11) satifies

(5.12)
$$t + ts^2(t) + 2G^{-,-}(t)G^{+,-}(t) + 2ts(t)G^{+,-}(t) + 2s(t)(G^{+,-}(t))^2 =$$
$$\frac{5}{2}t^3 - 4t^2 + \frac{3}{2}t - \frac{t(1-t)}{2}\sqrt{t^2 - 6t + 1} - \frac{t^2(1-t)^2}{\sqrt{t^2 - 6t + 1}} - \frac{t^2(1-t)^2(t-3)}{t^2 - 6t + 1}.$$

Let

(5.13) $$X = \sqrt{t^2 - 6t + 1}$$

and $Y = t - 3$. The denominator of $H^{+,-}(t)$ in (5.11) was calculated in (3.21), and is equal to $-\frac{1}{2}(X^2 + YX)$. We have

(5.14)

$$\frac{1}{X^2 + YX} = \frac{1}{Y}\left(\frac{1}{X} - \frac{1}{X+Y}\right) =$$

$$\frac{1}{Y}\left(\frac{1}{X} + \frac{Y-X}{X^2 - Y^2}\right) = \frac{1}{t-3}\left(\frac{1}{\sqrt{t^2 - 6t + 1}} + \frac{t - 3 - \sqrt{t^2 - 6t + 1}}{t^2 - 6t + 1 - (t-3)^2}\right) =$$

$$\frac{1}{t-3}\left(\frac{1}{\sqrt{t^2 - 6t + 1}} - \frac{t - 3 - \sqrt{t^2 - 6t + 1}}{8}\right) =$$

$$\frac{1}{(t-3)\sqrt{t^2 - 6t + 1}} - \frac{1}{8} + \frac{\sqrt{t^2 - 6t + 1}}{8(t-3)}.$$

From (5.11)–(5.14), we obtain

(5.15)

$$H^{+,-}(t) = -2\left(\frac{1}{(t-3)\sqrt{t^2 - 6t + 1}} - \frac{1}{8} + \frac{\sqrt{t^2 - 6t + 1}}{8(t-3)}\right) \times$$

$$\left((\frac{5}{2}t^3 - 4t^2 + \frac{3}{2}t) - \frac{t(1-t)}{2}\sqrt{t^2 - 6t + 1} - \frac{t^2(1-t)^2}{\sqrt{t^2 - 6t + 1}} - \frac{t^2(1-t)^2(t-3)}{t^2 - 6t + 1}\right).$$

Denote the three expressions in the first parenthetical factor in (5.15) by $\gamma_i, i = 1, 2, 3$, and denote the four expressions in the second parenthetical factor in (5.15) by $\beta_j, j = 1, 2, 3, 4$. We multiply out the right hand side of (5.15) in the order of the following double sum: $-2\sum_{j=1}^{4}\sum_{i=1}^{3}\gamma_i\beta_j$. We have $\gamma_2\beta_3 + \gamma_3\beta_4 = 0$. Also, after some algebra, one finds that $(\gamma_1\beta_3 + \gamma_2\beta_4) = \frac{t^2(1-t)^2}{8(t-3)}$, which cancels with the term $\gamma_3\beta_3$. Writing down the other seven terms in the order of the double sum above, and recalling the definition of X in (5.13), we obtain

(5.16)

$$H^{+,-}(t) = \frac{-5t^3 + 8t^2 - 3t}{t-3}\frac{1}{X} + \frac{5t^3 - 8t^2 + 3t}{8} + \frac{-5t^3 + 8t^2 - 3t}{8(t-3)}X +$$

$$\frac{t(1-t)}{t-3} - \frac{t(1-t)}{8}X + \frac{t(1-t)}{8(t-3)}(t^2 - 6t + 1) + 2t^2(1-t)^2\frac{1}{X^3}.$$

We have

(5.17)

$$\frac{t(1-t)}{t-3} + \frac{t(1-t)}{8(t-3)}(t^2 - 6t + 1) + \frac{5t^3 - 8t^2 + 3t}{8} =$$

$$\frac{1}{8}t(1-t)(t-3) - \frac{1}{8}(-3t + 8t^2 - 5t^3) = -\frac{1}{2}t^2(1-t)$$

and

$$(5.18) \qquad \frac{-5t^3 + 8t^2 - 3t}{8(t-3)} - \frac{t(1-t)}{8} = \frac{1}{2}t^2(1-t).$$

Now (5.3) for $H^{+,-}(t)$ follows from (5.16)-(5.18) and (5.13). We obtain (5.3) for $H^{-,-}(t)$ from (5.10), (5.3) for $H^{+,-}(t)$ and (2.5). $\qquad\square$

6. Proof of Theorem 2

We first proof part (i). Then we make minor additions to that proof to obtain part (ii). By (1.2), it suffices to consider $A_n^{+,-}$ for part (i). By (2.2) and (5.1), we have $E_n^{\mathrm{sep}}(A_n^{+,-})^2 = \frac{1}{2}C_n^{+,-} + \frac{1}{2}D_n^{+,-}$, and by Lemma 1, $D_n^{+,-} = C_n^{-,+} = C_n^{+,-}$. Therefore, $E_n^{\mathrm{sep}}(A_n^{+,-})^2 = C_n^{+,-}$. Thus $[t^n]H^{+,-}(t)$, the coefficient of t^n in the power series expansion of $H^{+,-}$, is equal to $s_n E_n^{\mathrm{sep}}(A_n^{+,-})^2$. As noted in Section 4, $[t^n]G^{+,-}(t)$ is equal to $s_n E_n^{\mathrm{sep}} A_n^{+,-}$. Thus

(6.1)

$$\mathrm{Var}_n^{\mathrm{sep}}(A_n^{+,-}) = E_n^{\mathrm{sep}}(A_n^{+,-})^2 - \left(E_n^{\mathrm{sep}} A_n^{+,-}\right)^2 = \frac{[t^n]H^{+,-}(t)}{s_n} - \left(\frac{[t^n]G^{+,-}(t)}{s_n}\right)^2.$$

We first consider the asymptotic behavior of $E_n^{\mathrm{sep}} A_n^{+,-} = \frac{[t^n]G^{+,-}(t)}{s_n}$. From 3.3 in Theorem 3 and from Proposition 1 in the appendix, we have

$$(6.2) \qquad [t^n]G^{+,-}(t) = a_{n-1} - a_{n-2},$$

where a_n satisfies (7.3). From (7.3),

(6.3)

$$a_{n-i} = \frac{(3 - 2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}}\left(\frac{1}{2^{\frac{5}{4}}}\frac{(3 - 2\sqrt{2})^i}{(n-i)^{\frac{1}{2}}} + \frac{3 - 4\sqrt{2}}{32 \cdot 2^{\frac{3}{4}}}\frac{(3 - 2\sqrt{2})^i}{(n-i)^{\frac{3}{2}}} + O\left(\frac{1}{n^{\frac{5}{2}}}\right)\right), i = 1, 2.$$

Motivated by (7.2), which gives the asymptotic behavior of s_n, we write (6.3) in the form

(6.4)

$$\sqrt{\pi}(3 - 2\sqrt{2})^{n-\frac{1}{2}}2^{\frac{3}{4}}n^{\frac{3}{2}}a_{n-i} = \frac{1}{3 - 2\sqrt{2}}\left(\frac{(3 - 2\sqrt{2})^i}{\sqrt{2}}n\left(\frac{n}{n-i}\right)^{\frac{1}{2}} + \right.$$

$$\left. \frac{(3 - 4\sqrt{2})(3 - 2\sqrt{2})^i}{32} + O(\frac{1}{n})\right), i = 1, 2.$$

Using (6.2), (6.4) and (7.2), we have

(6.5)
$$E_n^{\text{sep}} A_n^{+,-} = \frac{[t^n]G^{+,-}(t)}{s_n} = \left(1 + \frac{24 - 9\sqrt{2}}{32}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right)^{-1} \times$$

$$\frac{1}{3 - 2\sqrt{2}} \sum_{i=1}^{2} (-1)^{i-1} \left(\frac{(3 - 2\sqrt{2})^i}{\sqrt{2}} n\left(\frac{n}{n-i}\right)^{\frac{1}{2}} + \frac{(3 - 4\sqrt{2})(3 - 2\sqrt{2})^i}{32} + O\left(\frac{1}{n}\right)\right) =$$

$$\left(1 - \frac{24 - 9\sqrt{2}}{32}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right)\frac{1}{3 - 2\sqrt{2}} \times$$

$$\sum_{i=1}^{2} (-1)^{i-1} \left(\frac{(3 - 2\sqrt{2})^i}{\sqrt{2}} n\left(1 + \frac{i}{2n}\right) + \frac{(3 - 4\sqrt{2})(3 - 2\sqrt{2})^i}{32} + O\left(\frac{1}{n}\right)\right) =$$

$$(2 - \sqrt{2})n - \frac{1}{4}(3 - 2\sqrt{2}) + O\left(\frac{1}{n}\right),$$

where the last equality follows after some algebra. This proves (1.3). From (6.5) we obtain

$$(6.6) \quad \left(E_n^{\text{sep}} A_n^{+,-}\right)^2 = \left(\frac{[t^n]G^{+,-}(t)}{s_n}\right)^2 = (6 - 4\sqrt{2})n^2 - \frac{10 - 7\sqrt{2}}{2}n + O(1).$$

Now we turn to the asymptotic behavior of $E_n^{\text{sep}}(A_n^{+,-})^2 = \frac{[t^n]H^{+,-}(t)}{s_n}$. By (5.3) in Theorem 4, which gives the formula for $H^{+,-}(t)$, and Proposition 1 in the appendix, the leading order contribution in the coefficient of t^n in $H^{+,-}(t)$ comes from the leading order contribution in the coefficient of t^n in the term $2t^2(1-t)^2 \left(t^2 - 6t + 1\right)^{-\frac{3}{2}}$, while the next order contribution in the coefficient of t^n in $H^{+,-}(t)$ comes from the next order contribution in the coefficient of t^n in $2t^2(1-t)^2 \left(t^2 - 6t + 1\right)^{-\frac{3}{2}}$ and also from the leading order contribution in the coefficient of t^n in the term $\frac{-5t^3 + 8t^2 - 3t}{t-3} \left(t^2 - 6t + 1\right)^{-\frac{1}{2}}$.

Using the notation $r_1 = 3 - 2\sqrt{2}$, from (7.4) it follows that

the leading order contribution in the coefficient of t^n in

$$(6.7) \quad \frac{-5t^3 + 8t^2 - 3t}{t - 3} \left(t^2 - 6t + 1\right)^{-\frac{1}{2}} \text{ is}$$

$$\frac{(3 - 2\sqrt{2})^{-n+\frac{1}{2}}}{4 \cdot 2^{\frac{3}{4}}\sqrt{\pi}} \left(3(n-1)^{-\frac{1}{2}} - 8r_1(n-2)^{-\frac{1}{2}} + 5r_1^2(n-3)^{-\frac{1}{2}}\right) := I_1.$$

Writing $2t^2(1-t)^2 = 2t^2 - 4t^3 + 2t^4$, it follows from Proposition 1 that the two leading orders in the contribution to the coefficient of t^n in $2t^2(1 -$

$t)^2 \left(t^2 - 6t + 1\right)^{-\frac{3}{2}}$ are contained in $2\alpha_{n-2} - 4\alpha_{n-3} + 2\alpha_{n-4}$, where α_n satisfies (7.5). Thus, from (7.5),

the leading order contribution in the coefficient of t^n in

(6.8) $2t^2(1-t)^2 \left(t^2 - 6t + 1\right)^{-\frac{3}{2}}$ is given by

$$\frac{(3 - 2\sqrt{2})^{-n+\frac{1}{2}}}{4 \cdot 2^{\frac{3}{4}} \sqrt{\pi}} \left(2(n-2)^{\frac{1}{2}} - 4r_1(n-3)^{\frac{1}{2}} + 2r_1^2(n-4)^{\frac{1}{2}}\right) := I_2,$$

and

(6.9)
the second leading order contribution in the coefficient of t^n in

$2t^2(1-t)^2 \left(t^2 - 6t + 1\right)^{-\frac{3}{2}}$ is given by

$$\frac{(24 - 9\sqrt{2})(3 - 2\sqrt{2})^{-n+\frac{1}{2}}}{128 \cdot 2^{\frac{3}{4}} \sqrt{\pi}} \left(2(n-2)^{-\frac{1}{2}} - 4r_1(n-3)^{-\frac{1}{2}} + 2r_1^2(n-4)^{-\frac{1}{2}}\right) := I_3.$$

By (7.2),

the first two leading order terms in s_n are

(6.10) $$\frac{1}{\sqrt{\pi}} (3 - 2\sqrt{2})^{-n+\frac{1}{2}} \left(\frac{1}{2^{\frac{3}{4}}} n^{-\frac{3}{2}} + \frac{12\sqrt{2} - 9}{2^{\frac{1}{4}} \cdot 32} n^{-\frac{5}{2}}\right) := I_4.$$

From (6.7)-(6.10), we conclude that

the first two leading order terms in $E_n^{\text{sep}}(A_n^{+,-})^2 = \dfrac{[t^n]H^{+,-}(t)}{s_n}$

(6.11)
are contained in $\dfrac{I_1 + I_2 + I_3}{I_4}$ and are of the form $An^2 + Bn$.

From (6.7) and (6.10),
(6.12)

$$\frac{I_1}{I_4} = \frac{\frac{1}{4 \cdot 2^{\frac{3}{4}}} \left(\frac{3}{(n-1)^{\frac{1}{2}}} - \frac{8r_1}{(n-2)^{\frac{1}{2}}} + \frac{5r_1^2}{(n-3)^{\frac{1}{2}}}\right)}{\frac{1}{2^{\frac{3}{4}}} n^{-\frac{3}{2}} + \frac{12\sqrt{2} - 9}{2^{\frac{1}{4}} \cdot 32} n^{-\frac{5}{2}}} =$$

$$\frac{\frac{n}{4} \left(3(\frac{n}{n-1})^{\frac{1}{2}} - 8r_1(\frac{n}{n-2})^{\frac{1}{2}} + 5r_1^2(\frac{n}{n-3})^{\frac{1}{2}}\right)}{1 + \frac{(12\sqrt{2} - 9)\sqrt{2}}{32} \frac{1}{n}} = \frac{3 - 8r_1 + 5r_1^2}{4} n + O(1) =$$

$$\frac{3 - 8(3 - 2\sqrt{2}) + 5(3 - 2\sqrt{2})^2}{4} n + O(1) = (16 - 11\sqrt{2})n + O(1).$$

From (6.8) and (6.10),

(6.13)
$$\frac{I_2}{I_4} = \frac{\frac{1}{4\cdot 2^{\frac{3}{4}}}\left(2(n-2)^{\frac{1}{2}} - 4r_1(n-3)^{\frac{1}{2}} + 2r_1^2(n-4)^{\frac{1}{2}}\right)}{\frac{1}{2^{\frac{3}{4}}}n^{-\frac{3}{2}} + \frac{12\sqrt{2}-9}{2^{\frac{1}{4}}\cdot 32}n^{-\frac{5}{2}}} =$$

$$\frac{\frac{1}{2}(\frac{n-2}{n})^{\frac{1}{2}} - r_1(\frac{n-3}{n})^{\frac{1}{2}} + \frac{1}{2}r_1^2(\frac{n-4}{n})^{\frac{1}{2}}}{1 + \frac{(24-9\sqrt{2})}{32n}}n^2 =$$

$$\left(\frac{1}{2}(1-\frac{1}{n}) - r_1(1-\frac{3}{2n}) + \frac{1}{2}r_1^2(1-\frac{2}{n}) + O(\frac{1}{n^2})\right)\left(1 + \frac{9\sqrt{2}-24}{32n} + O(\frac{1}{n^2})\right)n^2 =$$

$$(6 - 4\sqrt{2})n^2 + (-\frac{79}{4} + \frac{219}{16}\sqrt{2})n + O(1).$$

From (6.9) and (6.10),

$$\frac{I_3}{I_4} = \frac{\frac{24-9\sqrt{2}}{128\cdot 2^{\frac{3}{4}}}\left(2(n-2)^{-\frac{1}{2}} - 4r_1(n-3)^{-\frac{1}{2}} + 2r_1^2(n-4)^{-\frac{1}{2}}\right)}{\frac{1}{2^{\frac{3}{4}}}n^{-\frac{3}{2}} + \frac{12\sqrt{2}-9}{32\cdot 2^{\frac{1}{4}}}n^{-\frac{5}{2}}} =$$

(6.14)
$$\frac{\frac{24-9\sqrt{2}}{64}\left((\frac{n}{n-2})^{\frac{1}{2}} - 2r_1(\frac{n}{n-3})^{\frac{1}{2}} + r_1^2(\frac{n}{n-4})^{\frac{1}{2}}\right)}{1 + \frac{24-9\sqrt{2}}{32n}}n =$$

$$\frac{24-9\sqrt{2}}{64}\left(1 - 2r_1 + r_1^2\right)n + O(1) = \frac{24-9\sqrt{2}}{64}(r_1-1)^2n + O(1) =$$

$$(\frac{24-9\sqrt{2}}{64})(2-2\sqrt{2})^2n + O(1) = \left(\frac{27}{4} - \frac{75}{16}\sqrt{2}\right)n + O(1).$$

From (6.11)-(6.14) we conclude that

(6.15)
$$E_n^{\text{sep}}(A_n^{+,-})^2 = \frac{[t^n]H^{+,-}(t)}{s_n} =$$

$$(6 - 4\sqrt{2})n^2 + \left(16 - 11\sqrt{2} - \frac{79}{4} + \frac{219}{16}\sqrt{2} + \frac{27}{4} - \frac{75}{16}\sqrt{2}\right)n + O(1) =$$

$$(6 - 4\sqrt{2})n^2 + (3 - 2\sqrt{2})n + O(1).$$

Using (6.15) with (6.6) and (6.1), we conclude that

(6.16)
$$\text{Var}_n^{\text{sep}}(A_n^{+,-}) \sim (3 - 2\sqrt{2} + \frac{10 - 7\sqrt{2}}{2})n = \frac{16 - 11\sqrt{2}}{2}n,$$

which proves (1.4).

We now consider part (ii). By (1.2), it suffices to consider $A_n^{-,-}$. We first consider the asymptotic behavior of $E_n^{\text{sep}}A_n^{-,-} = \frac{[t^n]G^{-,-}(t)}{s_n}$. From (3.3), we

have

$$(6.17) \qquad [t^n]G^{-,-}(t) = [t^n]G^{+,-}(t) - [t^n]\frac{1}{2}t(t^2 - 6t + 1)^{\frac{1}{2}}, \ n \geq 3.$$

By (7.1), we have

(6.18)
$$[t^n]\frac{1}{2}t(t^2 - 6t + 1)^{\frac{1}{2}} = \frac{1}{2}b_{n-1} = -\frac{1}{2^{\frac{3}{4}}\sqrt{\pi}}(3 - 2\sqrt{2})^{-n+\frac{3}{2}}\left(n^{-\frac{3}{2}} + O\left(n^{-\frac{5}{3}}\right)\right).$$

From (6.18) and (7.2), we obtain

$$(6.19) \qquad -\frac{[t^n]\frac{1}{2}t(t^2 - 6t + 1)^{\frac{1}{2}}}{s_n} = 3 - 2\sqrt{2} + O\left(\frac{1}{n}\right).$$

From (6.17), (6.5) and (6.19), we conclude that

$$(6.20) \quad E_n^{\mathrm{sep}}A_n^{-,-} = \frac{[t^n]G^{-,-}(t)}{s_n} = (2 - \sqrt{2})n + \frac{3}{4}(3 - 2\sqrt{2}) + O\left(\frac{1}{n}\right),$$

which proves (1.5).

We now turn to the variance. We have

(6.21)
$$\mathrm{Var}_n^{\mathrm{sep}}(A_n^{-,-}) = E_n^{\mathrm{sep}}(A_n^{-,-})^2 - \left(E_n^{\mathrm{sep}}A_n^{-,-}\right)^2 = \frac{[t^n]H^{-,-}(t)}{s_n} - \left(\frac{[t^n]G^{-,-}(t)}{s_n}\right)^2.$$

From (6.20), we have

$$(6.22) \ \ (E_n^{\mathrm{sep}}A_n^{-,-})^2 = \left(\frac{[t^n]G^{-,-}(t)}{s_n}\right)^2 = (6 - 4\sqrt{2})n^2 + \frac{30 - 21\sqrt{2}}{2}n + O(1).$$

From (5.3) and Proposition 1 in the appendix, we have

(6.23)
$$[t^n]H^{-,-}(t) = [t^n]H^{+,-}(t) + [t^n]2t^2(1-t)(t^2 - 6t + 1)^{-\frac{1}{2}} + \text{lower order terms in } n.$$

From (7.3), we have

$$[t^n]2t^2(1 - t)(t^2 - 6t + 1)^{-\frac{1}{2}} = 2(a_{n-2} - a_{n-3}) =$$

(6.24)
$$\frac{2^{\frac{1}{4}}}{\sqrt{\pi}}(3 - 2\sqrt{2})^{-n+\frac{3}{2}}(2 - \sqrt{2})\left(n^{-\frac{1}{2}} + O\left(n^{-\frac{3}{2}}\right)\right).$$

From (6.24) and (7.2), we obtain

(6.25)
$$\frac{[t^n]2t^2(1 - t)(t^2 - 6t + 1)^{-\frac{1}{2}}}{s_n} = 2(3 - 2\sqrt{2})(2 - \sqrt{2})n + O(1) = (20 - 14\sqrt{2})n + O(1).$$

From (6.23), (6.25) and (6.15), we obtain

(6.26)
$$\frac{[t^n]H^{-,-}(t)}{s_n} = (6 - 4\sqrt{2})n^2 + (3 - 2\sqrt{2})n + (20 - 14\sqrt{2})n + O(1) =$$
$$(6 - 4\sqrt{2})n^2 + (23 - 16\sqrt{2})n + O(1).$$

From (6.21), (6.22) and (6.26), we conclude that

$$\mathrm{Var}_n^{\mathrm{sep}}(A_n^{-,-}) \sim (23 - 16\sqrt{2})n - \frac{30 - 21\sqrt{2}}{2}n = \frac{16 - 11\sqrt{2}}{2}n,$$

which proves (1.6). □

7. APPENDIX

We present a proposition that contains the asymptotic analysis needed in the proofs of Theorems 1 and 2. For a function $f(t)$ given by a power series expansion $f(t) = \sum_{n=0}^{\infty} f_n t^n$, we use the notation $[t^n]f(t) = f_n$.

Proposition 1.

(7.1)
$$b_n := [t^n]\left(t^2 - 6t + 1\right)^{\frac{1}{2}} = -\frac{1}{\sqrt{\pi}}(3 - 2\sqrt{2})^{-n+\frac{1}{2}} \times$$
$$\left(2^{\frac{1}{4}}n^{-\frac{3}{2}} + \frac{12\sqrt{2} - 9}{16 \cdot 2^{\frac{1}{4}}}n^{-\frac{5}{2}} + O(n^{-\frac{7}{2}})\right).$$

(7.2)
$$s_n = [t^n]\left(\frac{1 - t - (t^2 - 6t + 1)^{\frac{1}{2}}}{2}\right) = \frac{1}{\sqrt{\pi}}(3 - 2\sqrt{2})^{-n+\frac{1}{2}} \times$$
$$\left(\frac{1}{2^{\frac{3}{4}}}n^{-\frac{3}{2}} + \frac{12\sqrt{2} - 9}{32 \cdot 2^{\frac{1}{4}}}n^{-\frac{5}{2}} + O(n^{-\frac{7}{2}})\right).$$

(7.3)
$$a_n := [t^n]\left(t^2 - 6t + 1\right)^{-\frac{1}{2}} = \frac{(3 - 2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}} \times$$
$$\left(\frac{1}{2^{\frac{5}{4}}}n^{-\frac{1}{2}} + \frac{3 - 4\sqrt{2}}{32 \cdot 2^{\frac{3}{4}}}n^{-\frac{3}{2}} + O(n^{-\frac{5}{2}})\right).$$

(7.4) $\quad [t^n]\dfrac{\left(t^2 - 6t + 1\right)^{-\frac{1}{2}}}{t - 3} = -\dfrac{(3 - 2\sqrt{2})^{-n-\frac{1}{2}}}{\sqrt{\pi}}\left(\dfrac{1}{4 \cdot 2^{\frac{3}{4}}}n^{-\frac{1}{2}} + O\left(n^{-\frac{3}{2}}\right)\right).$

$$(7.5) \quad \alpha_n := [t^n] \left(t^2 - 6t + 1\right)^{-\frac{3}{2}} = \frac{(3 - 2\sqrt{2})^{-n - \frac{3}{2}}}{\sqrt{\pi}} \times$$

$$\left(\frac{1}{4 \cdot 2^{\frac{3}{4}}} n^{\frac{1}{2}} + \frac{24 - 9\sqrt{2}}{128 \cdot 2^{\frac{3}{4}}} n^{-\frac{1}{2}} + O(n^{-\frac{3}{2}}) \right).$$

Proof. For $\alpha \in \mathbb{C}$, write

$$(7.6) \quad \left(t^2 - 6t + 1\right)^{-\alpha} = (1 - \frac{t}{r_1})^{-\alpha}(1 - \frac{t}{r_2})^{-\alpha},$$

where

$$r_1 = 3 - 2\sqrt{2}, \quad r_2 = 3 + 2\sqrt{2}.$$

By [5, p. 381], one has

$$(7.7)$$
$$[t^n] \left(1 - \frac{t}{r_1}\right)^{-\alpha} = \frac{1}{r_1^n} \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \frac{\alpha(\alpha-1)}{2n} + O(\frac{1}{n^2})\right), \text{ for } \alpha \in \mathbb{C} - \mathbb{Z}_{\leq 0}.$$

Noting that $r_1 r_2 = 1$, a direct calculation of the first two Taylor coefficients of $\left(1 - \frac{t}{r_2}\right)^{-\alpha}$ about $t = r_1$ yields

$$(7.8)$$
$$\left(1 - \frac{t}{r_2}\right)^{-\alpha} = \frac{1}{r_1^\alpha(r_2 - r_1)^\alpha} - \frac{\alpha}{r_1^{\alpha-1}(r_2 - r_1)^{\alpha+1}} \left(1 - \frac{t}{r_1}\right) + O\left(\left(1 - \frac{t}{r_1}\right)^2\right).$$

From (7.6) and (7.8) we obtain

$$(7.9)$$
$$\left(t^2 - 6t + 1\right)^{-\alpha} = \frac{1}{r_1^\alpha(r_2 - r_1)^\alpha}(1 - \frac{t}{r_1})^{-\alpha} - \frac{\alpha}{r_1^{\alpha-1}(r_2 - r_1)^{\alpha+1}} \left(1 - \frac{t}{r_1}\right)^{1-\alpha} +$$

$$O\left((1 - \frac{t}{r_1})^{2-\alpha}\right).$$

It then follows by the transfer theorem [5, Theorem VI.3, p. 390, Example VI.2, p. 395] that the asymptotic behavior of $[t^n](t^2 - 6t + 1)^{-\alpha}$ can be read off term by term from (7.7) and (7.9). Grouping the appropriate terms yields

$$(7.10)$$
$$[t^n](t^2 - 6t + 1)^{-\alpha} = \frac{1}{r_1^n} \left(\frac{1}{\Gamma(\alpha)r_1^\alpha(r_2 - r_1)^\alpha} n^{\alpha-1} + \right.$$

$$\left. \frac{\alpha}{r_1^\alpha(r_2 - r_1)^\alpha} \left(\frac{\alpha-1}{2\Gamma(\alpha)} - \frac{r_1}{(r_2 - r_1)\Gamma(\alpha-1)} \right) n^{\alpha-2} + O(n^{\alpha-3}) \right), \quad \alpha \in \mathbb{C} - Z_{\leq 0}.$$

Substituting $r_1 = 3 - 2\sqrt{2}$ and $r_2 = 3 + 2\sqrt{2}$ in (7.10), and choosing $\alpha = -\frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$ respectively, yields (7.1), (7.3) and (7.5) respectively. Of course, (7.2) follows immediately from (7.1). From the transfer theorem it follows that the leading order term in the asymptotic behavior of $[t^n]g(t)\left(t^2 - 6t + 1\right)^{-\frac{1}{2}}$, where $g(t)$ is analytic in a disk, centered at the origin, of radius larger than r_1 is $g(r_1)$ times the leading order term in the asymptotic behavior of $[t^n]\left(t^2 - 6t + 1\right)^{-\frac{1}{2}}$. Choosing $g(t) = \frac{1}{t-3}$, (7.4) follows from (7.3). $\qquad\square$

References

[1] Avis, D. and Newborn, M. *On pop-stacks in series*, Utilitas Mathematica **19** (1981), 129-140.

[2] Bóna, M., *Combinatorics of Permutations*, Discrete Math. Appl. (Boca Raton) CRC Press, Boca Raton, FL, (2012).

[3] Bose, P., Buss, J. and Lubiw, A., *Pattern matching for permutations*, Inform. Process. Lett. **65** (1998), 277-283.

[4] Firro, G., Mansour, T. and Wilson, M. C., *Longest alternating subsequences in pattern-restricted permutations*, Electron. J. Combin. **14** (2007), no.1, Research Paper 34, 17 pp.

[5] Flajolet, P. and Sedgewick, R., *Analytic Combinatorics*, Cambridge University Press, Cambridge, (2009).

[6] Ghys, É., *A Singular Mathematical Promenade*, ENS Éditions, Lyon, (2017).

[7] Knuth, D.E., *The Art of Computer Programming. Vol. 1: Fundamental Algorithms*, Second printing Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont (1969).

[8] Shapiro, L. and Stephens, A. B., *Bootstrap percolation, the Schröder numbers, and the n-kings problem*, SIAM J.Discrete Math. **4** (1991), 275-280.

[9] Stanley, R., *Longest alternating subsequences of permutations*, Michigan Math. J. 57 (2008), 675-687.

Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

Email address: pinsky@technion.ac.il

URL: https://pinsky.net.technion.ac.il/