A LOCAL VERSION OF THE KEMENEY CONSTANT PHENOMENON AND RANDOM SEARCH OPTIMIZATION FOR ONE-DIMENSIONAL DIFFUSIONS

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ABSTRACT. Consider a positive recurrent, one-dimensional diffusion process. Let τ_y denote the first time it hits y and denote by μ_{inv} and $\mu_{\text{inv}}(x)$ its invariant probability distribution and density. Consider the quantity $K_{\mu_{\text{inv}}}(x) := \int_{-\infty}^{\infty}(E_x\tau_y)\mu_{\text{inv}}(y)dy$, which is the expected time starting from $x \in \mathbb{R}$ until the diffusion process hits a point that is randomly chosen according to the invariant measure of the process. In [7], it was shown that this quantity is independent of x. This is a generalization of the classical Kemeny constant result for positive recurrent, irreducible finite state Markov chains. The fact that the above quantity is independent of x is only nontrivial if the quantity is finite, and it was also shown in [7] that the quantity is finite if and only if $\pm\infty$ are entrance boundaries for the diffusion. The entrance boundary requirement is a very strong restriction on a positive recurrent diffusion. In this paper, we prove a result of the same flavor that is nontrivial for all positive recurrent diffusions. Namely, let $[c, d]$ be an interval and let μ be any probability measure on $\mathbb{R}$ that satisfies $\mu((-\infty, x]) = \mu_{\text{inv}}(-\infty, x])$, for $x \in [c, d]$. Let $K_\mu(x) = \int_{-\infty}^{\infty}(E_x\tau_y)\mu(dy)$. Then $K_\mu(x)$ is constant for $x \in [c, d]$. The measure μ can always be chosen so that the constant is finite; in particular, it will always be finite if μ is chosen to have compact support. We also study the local minima and maxima of K_μ for general probability measures μ without any reference to μ_{inv}. The quantity $K_\mu(x)$ may be thought of as the expected time for a diffusive searcher starting from x to locate a target that has been randomly positioned according to the distribution μ.

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1. Introduction and Statement of Results

Before defining a local version of the Kemeny constant for one-dimensional diffusion processes, we recall some results concerning the classical Kemeny constant. Let $\{X_n\}_{n=0}^{\infty}$ be an irreducible, discrete time Markov chain on a finite state space S, and denote its invariant probability measure by μ. For $j \in S$, let $\tau_j = \inf\{n \geq 0 : X_n = j\}$ denote the hitting time of j. Denoting expectations for the process starting from $i \in S$ by E_i, consider the quantity $\sum_{j\in S} \mu_j E_i \tau_j$, which is the expected time starting from the state i until the Markov chain hits a point that is randomly chosen according to the invariant measure of the process. In their book on Markov chains [4], Kemeny and Snell showed that the above quantity is independent of the initial state i; this quantity has become known as *Kemeny's constant.* See also, the nice argument by Doyle [2].

In [1], the Kemeny constant phenomenon was studied for irreducible, positive recurrent, discrete time and continuous time Markov chains on a denumerably infinite state space S. It was shown that the quantity $\sum_{j\in S} \mu_j E_i \tau_j$ is either infinite for all $i \in S$, or else is finite and independent of i. The authors conjectured that this quantity is always infinite in the discrete time setting, and they proved this in the case of discrete time birth and death chains on $\{0, 1, \cdots\}$. In the case of continuous time birth and death chains on $\{0, 1, \cdots\}$, they proved that the Kemeny constant is finite if and only if ∞ is an *entrance boundary* for the process.

In [7], we considered the corresponding problem in the context of positive recurrent, one-dimensional Markov diffusion processes $X(t)$ on $\mathbb{R}$. The generator of $X(t)$ was assumed to be of the form

$$L = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \tag{1.1}$$

where a is continuous and positive and b is locally bounded and measurable. Denote probabilities and expectations for the Markov process $X(\cdot)$ starting from $x \in \mathbb{R}$ by P_x and E_x. For $y \in \mathbb{R}$, let $\tau_y = \inf\{t \geq 0 : X(t) = y\}$ denote the first hitting time of y. As is well known [5], positive recurrence is equivalent to the condition

$$\int_{-\infty}^{\infty} \frac{1}{a(x)} \exp\left(2\int_0^x \frac{b(t)}{a(t)}dt\right) dx < \infty, \tag{1.2}$$

and when the diffusion is positive recurrent, its invariant distribution μ_{inv} has density, which we call $\mu_{\text{inv}}(x)$, given by

$$\mu_{\text{inv}}(x) = c_0 \frac{1}{a(x)} \exp\left(2\int_0^x \frac{b(t)}{a(t)}dt\right), \ x \in \mathbb{R}, \tag{1.3}$$

where c_0 is the normalization constant. The quantity

$$K_{\mu_{\text{inv}}}(x) := \int_{-\infty}^{\infty} (E_x\tau_y)\mu_{\text{inv}}(y)dy, \ x \in \mathbb{R},$$

is the expected time starting from $x \in \mathbb{R}$ until the diffusion process hits a point that is randomly chosen according to the invariant measure of the process. In [7] it was proven that $K_{\mu_{\text{inv}}}(x)$ is independent of x.

However, this result is only interesting if this Kemeny constant is finite. It was shown in [7] that this Kemeny constant is finite if and only $\pm\infty$ are entrance boundaries for the diffusion, analogous to the situation for continuous time birth and death processes mentioned above. For use in section 3, we note that the proof in [7] shows that $\int^{\infty}(E_x\tau_y)\mu_{\text{inv}}(y)dy$ is finite if and only if $+\infty$ is an entrance boundary, and $\int_{-\infty}(E_x\tau_y)\mu_{\text{inv}}(y)dy$ is finite if and only if $-\infty$ is an entrance boundary.

The boundary $+\infty$ is called an *entrance boundary* if $\lim_{x\to\infty} P_x(\tau_y < t) > 0$, for some $y \in \mathbb{R}$ and some $t > 0$ (or equivalently, for all $y \in \mathbb{R}$ and all $t > 0$). A parallel definition is given for the boundary $-\infty$. For the precise criterion for an entrance boundary, in terms of the coefficients a and b of the generator L of the process, see [7]. Heuristically, in order for $\pm\infty$ to be entrance boundaries, the drift $b(x)$ has to push the process inward sufficiently fast from $\pm\infty$. In particular, we note the following comparison between the requirement for positive recurrence and the requirement for an entrance boundary:

(1.4)

$$\begin{aligned}
&\text{if } a(x) \text{ is bounded and bounded away from 0 and } b(x) = -\text{sgn}(x)c|x|^l,\\
&\text{for large } |x|, \quad \text{with } c > 0 \ \text{ and } l \in \mathbb{R}, \quad \text{then}\\
&i. \pm\infty \ \text{ are entrance boundaries if and only } l > 1;\\
&ii. \text{ the diffusion is positive recurrent if and only if either } l > -1\\
&\text{or } l = -1 \text{ and } c > \frac{1}{2}.
\end{aligned}$$

We point out that if one considers a diffusion process on a finite interval, with reflection at the boundaries, in which case the diffusion is always positive recurrent, one can use the techniques in [7] to show that the corresponding Kemeny quantity is constant and always finite, analogous to the case of a Markov chain on a finite state space.

We the above background at hand, we turn to the problem studied in this paper. Let $y \in \mathbb{R}$ denote an unknown random target with known distribution $\mu \in \mathcal{P}(\mathbb{R})$, the space of probability measures on $\mathbb{R}$. A diffusive searcher $X(t)$, generated by L as above, starts out from some point $x \in \mathbb{R}$. The expected time to locate the target is given by

$$K_\mu(x) = \int_{-\infty}^{\infty} (E_x \tau_y)\mu(dy), \ x \in \mathbb{R}. \tag{1.5}$$

We only consider distributions μ for which $K_\mu(x)$ is finite. (If it is finite for one x, then it is finite for all x.) It is then natural to consider where K_μ attains its minimum, and perhaps also, its maximum. Note, however, that it follows from the definition of an entrance boundary that if $+\infty$ $(-\infty)$ is not an entrance boundary, then $\lim_{x\to+\infty} K_\mu(x) = \infty$ $(\lim_{x\to-\infty} K_\mu(x) = \infty)$. Of course, if $\mu = \mu_{\text{inv}}$, which under our assumption that K_μ is finite is allowed only if $\pm\infty$ are entrance boundaries, then by the result from [7] noted above, $K_\mu(x)$ is constant.

Before continuing in the exposition, we note some background with regard to this random search problem. In [6], this setup was studied from a different angle; the diffusion coefficient a was assumed to be constant, the starting point of the searcher was fixed at $x = 0$, and the problem was to find for each target distribution μ, the drift b that minimizes $K_\mu(0)$. Note that if one lets the diffusive searcher be standard Brownian motion, then the process is not positive recurrent and consequently K_μ is infinite for all choices of non-degenerate probability measures μ. In order to overcome this issue, for a Brownian searcher one considers the problem with resetting; namely, at an exponential time, the searcher jumps back to its starting point and begins its search anew; this ensures the finiteness of the expected search time. This has been a much studied topic over the past 15 years; see [3] for a review article and many references.

Our interest in the minimum and maxium of K_μ led us to what we call a local version of the Kemeny constant phenomenon. Here is the result.

Theorem 1. *Let L, as in* (1.1)*, be the generator of a positive recurrent diffusion on $\mathbb{R}$; that is, assume* (1.2) *holds. Denote by μ_{inv} the invariant distribution with invariant density $\mu_{inv}(x)$ as in* (1.3)*. Let $-\infty < c < d < \infty$. Let $\mu \in \mathcal{P}(\mathbb{R})$ be such that K_μ is finite, where K_μ is as in* (1.5)*. Assume that*

$$\mu\left((-\infty, x]\right) = \mu_{inv}\left((-\infty, x]\right), \text{ for all } x \in [c, d]. \tag{1.6}$$

Then $K_\mu(x)$ is constant for $x \in [c, d]$.

Remark 1. As noted above, the fact that $K_{\mu_{\rm inv}}(x)$ is constant is only non-trivial if it is finite, and it is finite if and only if $\pm\infty$ are entrance boundaries for the diffusion process $X(t)$. This requirement is a very restrictive one on a positive recurrent diffusion, as (1.4) demonstrates. Theorem 1, on the other hand, is nontrivial for all positive recurrent diffusions. Indeed, fix c, d as in the theorem. Let ν_- and ν_+ be sub-probability measures, supported respectively on $(-\infty, c]$ and on $[d, \infty)$, that satisfy $\nu_-\left((-\infty, c]\right) = \mu_{\rm inv}\left((-\infty, c]\right)$ and $\nu_+\left([d, \infty)\right) = \mu_{\rm inv}\left([d, \infty)\right)$. Assume also that $\int_{-\infty}^{\infty}(E_x\tau_y)\nu_\pm(dy) < \infty$. (This last requirement is always fulfilled, for example, if $\nu_\pm$ are chosen to have compact support.) Then the theorem applies to the distribution μ defined by

$$\mu(A) = \begin{cases} \nu_-(A), \ A \subset (-\infty, c]; \\ \mu_{\rm inv}(A), \ A \subset (c, d); \\ \nu_+(A), \ A \subset [d, \infty). \end{cases}$$

Remark 2. If K_μ is finite and satisfies (1.6), then without reference to the conclusion of Theorem 1, we have

(1.7)

$$K_\mu(x) = \int_{-\infty}^{c} (E_x\tau_y)\mu(dy) + \int_c^d (E_x\tau_y)\mu_{\rm inv}(y)dy + \int_d^\infty (E_x\tau_y)\mu(dy) =$$
$$\int_{-\infty}^{c} (E_x\tau_c + E_c\tau_y)\,\mu(dy) + \int_c^d (E_x\tau_y)\mu_{\rm inv}(y)dy + \int_d^\infty (E_x\tau_d + E_d\tau_y)\,\mu(dy) =$$
$$\Big[\mu_{\rm inv}\left((-\infty,c]\right)E_x\tau_c + \int_c^d (E_x\tau_y)\mu_{\rm inv}(y)dy + \mu_{\rm inv}\left((d,\infty)\right)E_x\tau_d\Big] +$$
$$\Big[\int_{-\infty}^{c} (E_c\tau_y)\mu(dy) + \int_d^\infty (E_d\tau_y)\mu(dy)\Big] =$$
$$F_1(x) + F_2(\mu), \text{ for } x \in [c,d],$$

where $F_1(x)$, which is independent of μ, denotes the third line in the above equation and $F_2(\mu)$, which is independent of x, denotes the fourth line in the above equation. This shows that for any two probability measures μ_1, μ_2 satisfying (1.6), and for which K_{μ_1} and K_{μ_2} are finite, the difference $K_{\mu_2}(x) - K_{\mu_1}(x)$ is constant over $x \in [c,d]$. Thus, (1.7) along with the Kemeny result in [7] prove Theorem 1 in the case that $K_{\mu_{\rm inv}}$ is finite; indeed, let $\mu = \mu_1$ and $\mu_{\rm inv} = \mu_2$. We also note that (1.7) along with Theorem 1 give the following corollary to Theorem 1.

Corollary 1. *Let $-\infty < c < d < \infty$. The quantity*

$$\mu_{inv}\left((-\infty,c]\right)E_x\tau_c + \int_c^d (E_x\tau_y)\mu_{inv}(y)dy + \mu_{inv}\left((d,\infty)\right)E_x\tau_d$$

is independent of $x \in [c,d]$.

We now return to the question of locating the local minima and maxima of $K_\mu(x)$ for general μ.

Theorem 2. *Let L as in* (1.1) *be the generator of a positive recurrent diffusion on $\mathbb{R}$; that is, assume* (1.2) *holds. Denote by μ_{inv} the invariant distribution with invariant density $\mu_{inv}(x)$ as in* (1.3)*. Let μ be a probability measure on $\mathbb{R}$ for which $K_\mu(x)$, defined in* (1.5)*, is finite.*

i.

$$K'_\mu(x) = \frac{2}{c_0}\exp\left(-2\int_0^x \frac{b(t)}{a(t)}dt\right)\big(\mu((-\infty,x]) - \mu_{inv}((-\infty,x])\big), \tag{1.8}$$

where c_0 *is as in* (1.3)*. In particular,* $K'_\mu(x) = 0$ *if and only if* $\mu((-\infty, x]) = \mu_{inv}((-\infty, x])$.
ii. Assume that μ *possesses a continuous density* $\mu(x)$*. If* $K'_\mu(x_0) = 0$*, then* x_0 *is a local minimum for* K_μ *if* $\mu(x_0) > \mu_{inv}(x_0)$*, and* x_0 *is a local maximum for* K_μ *if* $\mu(x_0) < \mu_{inv}(x_0)$*. If* $\mu(x_0) = \mu_{inv}(x_0)$*, and* $\mu(x) - \mu_{inv}(x)$ *changes sign at* $x = x_0$*, then* x_0 *is neither a maximum nor a minimum for* K_μ*, while if* $\mu(x) - \mu_{inv}(x)$ *is positive (negative) for* $|x - x_0|$ *sufficiently small, then* x_0 *is a minimum (maximum).*

Note that Theorem 1 follows from part (i) of Theorem 2. We prove Theorem 2 in section 2. In section 3 we illustrate Theorem 2 with some examples.

2. Proof of Theorem 2

Proof of part (i). We have the following explicit expression for the expected hitting time:

$$E_x\tau_y = \begin{cases} 2\int_y^x dz \exp(-2\int_0^z \frac{b(t)}{a(t)}dt) \int_z^\infty dw \frac{1}{a(w)} \exp(2\int_0^w \frac{b(t)}{a(t)}dt), & -\infty < y < x; \\ 2\int_x^y dz \exp(-2\int_0^z \frac{b(t)}{a(t)}dt) \int_{-\infty}^z dw \frac{1}{a(w)} \exp(2\int_0^w \frac{b(t)}{a(t)}dt), & x < y < \infty. \end{cases} \tag{2.1}$$

For a derivation, see for example the proof of Proposition 2 in [6] (where $a(x)$ is a constant and denoted by D). Thus,

$$\begin{aligned} K_\mu(x) = &\int_{-\infty}^x \left(2\int_y^x dz \exp(-2\int_0^z \frac{b(t)}{a(t)}dt) \int_z^\infty dw \frac{1}{a(w)} \exp(2\int_0^w \frac{b(t)}{a(t)}dt)\right) \mu(dy) + \\ &\int_x^\infty \left(2\int_x^y dz \exp(-2\int_0^z \frac{b(t)}{a(t)}dt) \int_{-\infty}^z dw \frac{1}{a(w)} \exp(2\int_0^w \frac{b(t)}{a(t)}dt)\right) \mu(dy). \end{aligned} \tag{2.2}$$

Denote the first summand in (2.2) by $I(x)$ and the second summand there by $II(x)$. Denote the integrand in $I(x)$ by $G(x, y)$, so $I(x) = \int_{-\infty}^x G(x, y)\mu(dy)$. Recall that μ is an arbitrary probability measure on $\mathbb{R}$. However, since $G(x, x) = 0$, we do not need to be careful about the possibility of μ having an atom at the upper limit x of the integral. For $h > 0$, We write

$$\begin{aligned} &\frac{1}{h}\left(I(x+h) - I(x)\right) = \\ &\int_{-\infty}^{x+h} \frac{1}{h}\left(G(x+h, y) - G(x, y)\right)\mu(dy) + \frac{1}{h}\int_x^{x+h} G(x, y)\mu(dy). \end{aligned} \tag{2.3}$$

Using the mean value theorem, we write the first summand on the right hand side of (2.3) as

$$\begin{aligned}&\int_{-\infty}^{x+h}\frac{1}{h}\left(G(x+h,y)-G(x,y)\right)\mu(dy)=\\&\int_{-\infty}^{x+h}\frac{\partial G}{\partial x}(x+h^*(x,y),y)\mu(dy),\ \text{where } 0<h^*(x,y)<h.\end{aligned}\tag{2.4}$$

Note that $\frac{\partial G}{\partial x}(z,y)=2\exp(-2\int_0^z\frac{b(t)}{a(t)}dt)\int_z^\infty dw\frac{1}{a(w)}\exp(2\int_0^w\frac{b(t)}{a(t)}dt)$, and is thus independent of y. Thus, from (2.4) we have

$$\begin{aligned}&\lim_{h\to0^+}\int_{-\infty}^{x+h}\frac{1}{h}\left(G(x+h,y)-G(x,y)\right)\mu(dy)=\\&\left(2\exp(-2\int_0^x\frac{b(t)}{a(t)}dt)\int_x^\infty dw\frac{1}{a(w)}\exp(2\int_0^w\frac{b(t)}{a(t)}dt)\right)\mu((-\infty,x]).\end{aligned}\tag{2.5}$$

Since $G(x,y)$ is continuously differentiable in y, and since $G(x,x)=0$, we have $|G(x,y)|=O\left((y-x|\right)$. Using this estimate and again the fact that $G(x,x)=0$, the second summand on the right hand side of (2.3) satisfies

$$|\frac{1}{h}\int_x^{x+h}G(x,y)\mu(dy)|=\frac{O(h)}{h}\mu((x,x+h]).\tag{2.6}$$

From (2.6) it follows that

$$\lim_{h\to0^+}\frac{1}{h}\int_x^{x+h}G(x,y)\mu(dy)=0.\tag{2.7}$$

We conclude from (2.3), (2.5) and (2.7) along with similar calculations when $h\to0^-$ that

$$\begin{aligned}&\lim_{h\to0}\frac{1}{h}\left(I(x+h)-I(x)\right)=\\&\left(2\exp(-2\int_0^x\frac{b(t)}{a(t)}dt)\int_x^\infty dw\frac{1}{a(w)}\exp(2\int_0^w\frac{b(t)}{a(t)}dt)\right)\mu((-\infty,x]).\end{aligned}\tag{2.8}$$

A similar calculation for $II(x)$ shows that

$$\begin{aligned}&\lim_{h\to0}\frac{1}{h}\left(II(x+h)-II(x)\right)=\\&-\left(2\exp(-2\int_0^x\frac{b(t)}{a(t)}dt)\int_{-\infty}^x dw\frac{1}{a(w)}\exp(2\int_0^w\frac{b(t)}{a(t)}dt)\right)\mu([x,\infty)).\end{aligned}\tag{2.9}$$

From (2.3), (2.8) and (2.9), we obtain

(2.10)

$$K'_\mu(x) = \Big(2\exp(-2\int_0^x \frac{b(t)}{a(t)}dt)\int_x^\infty dw\frac{1}{a(w)}\exp(2\int_0^w \frac{b(t)}{a(t)}dt)\Big)\mu((-\infty,x])-$$
$$\Big(2\exp(-2\int_0^x \frac{b(t)}{a(t)}dt)\int_{-\infty}^x dw\frac{1}{a(w)}\exp(2\int_0^w \frac{b(t)}{a(t)}dt)\Big)\mu([x,\infty)).$$

Recalling the formula in (1.3) for the invariant density $\mu_{\text{inv}}(x)$, we conclude from (2.10) that

(2.11)

$$K'_\mu(x) =$$
$$\frac{2}{c_0}\exp\Big(-2\int_0^x \frac{b(t)}{a(t)}dt\Big)\big(\mu_{\text{inv}}([x,\infty))\mu((-\infty,x]) - \mu_{\text{inv}}((-\infty,x])\mu([x,\infty))\big).$$

The right hand side of (2.11) is equivalent to the right hand side of (1.8).

Proof of part (ii). We are now assuming that μ has a continuous density $\mu(x)$. Differentiating (1.8), we have

$$K''_\mu(x) = \frac{2}{c_0}\exp\Big(-2\int_0^x \frac{b(t)}{a(t)}dt\Big)\times$$
$$\Big(-2\frac{b(x)}{a(x)}\big(\mu((-\infty,x]) - \mu_{\text{inv}}((-\infty,x])\big) + \mu(x) - \mu_{\text{inv}}(x)\Big). \tag{2.12}$$

From (1.8), it follows that if $K'_\mu(x_0) = 0$, then the sign of $K''_\mu(x_0)$ is the same as that of $\mu(x_0) - \mu_{\text{inv}}(x_0)$. This proves part (ii) accept for the case that $\mu(x_0) = \mu_{\text{inv}}(x_0)$. Consider now the case that $\mu(x_0) = \mu_{\text{inv}}(x_0)$. If $\mu(x) > \mu_{\text{inv}}(x)$ when $x \neq x_0$ and $|x - x_0|$ is sufficiently small, then by (1.8), it follows that for $x \neq x_0$ and $|x - x_0|$ is sufficiently small, $K'_\mu(x)$ is negative when $x < x_0$ and positive when $x > x_0$, and consequently x_0 is a minimum for K_μ. If $\mu(x) < \mu_{\text{inv}}(x)$ when $x \neq x_0$ and $|x - x_0|$ is sufficiently small, then similar reasoning shows that x_0 is a maximum for K_μ. Finally, if $\mu(x) - \mu_{\text{inv}}(x)$ changes sign at x_0, then similar reasoning shows that x_0 is neither a maximum nor a minimum for K_μ. □

3. Some Examples

We illustrate Theorem 2 by considering a generic set of examples and then concretize this with specific examples. Our generic assumption is that μ has a density $\mu(x)$ and the graphs of the densities $\mu_{\text{inv}}(x)$ and $\mu(x)$ intersect one another a finite number of times. It then follows from Theorem 2 that

K_μ is monotone in a neighborhood of $+\infty$ and in a neighborhood of $-\infty$; consequently, $K_\mu(\pm\infty) = \lim_{x\to\pm\infty} K_\mu(x) \in (0,\infty]$ exists.

We divide the above assumption into the following three situations:

I. $\mu(x) < \mu_{\text{inv}}(x)$ for sufficiently large $|x|$;

II. $\mu_{\text{inv}}(x) < \mu(x)$ for sufficiently large $|x|$;

III-a. $\mu(x) < \mu_{\text{inv}}(x)$ for x in a neighborhood of $+\infty$ and $\mu_{\text{inv}}(x) < \mu(x)$ for x in a neighborhood of $-\infty$;

III-b. $\mu_{\text{inv}}(x) < \mu(x)$ for x in a neighborhood of $+\infty$ and $\mu(x) < \mu_{\text{inv}}(x)$ for x in a neighborhood of $-\infty$.

As in the theorem, we always assume that $K_\mu(x) < \infty$, for $x \in \mathbb{R}$. Thus by the theorem from [7] and its proof, as noted in section 1, if neither of $\pm\infty$ is an entrance boundary, then μ must satisfy (I). If $+\infty$ is an entrance boundary and $-\infty$ is not one, then μ must satisfy either (I) or (III-b), while if $+\infty$ is not an entrance boundary and $-\infty$ is one, then μ must satisfy either (I) or (III-a). If both $\pm\infty$ are entrance boundaries, then for any of the three conditions above, there exist appropriate μ satisfying that condition. By straight forward analysis, which we omit, Theorem 2 allows for the following conclusions under each of the above situations:

Theorem 2 under condition (I). Then $|\{x \in \mathbb{R} : \mu(x) = \mu_{\text{inv}}(x)\}| = 2n$, for some $n \in \mathbb{N}$. The function $K_\mu(x)$, $x \in \mathbb{R}$, possesses k local minima and $k-1$ local maxima, for some $k \in [n]$. In addition, $\pm\infty$ are local maxima. In particular, if $+\infty$ ($-\infty$) is not an entrance boundary, then $K_\mu(+\infty) = \infty$ ($K_\mu(-\infty) = \infty$). If $k = 1$, then then the unique local minimum is the global minimum for K_μ, and the supremum is at $+\infty$, $-\infty$ or both.

Theorem 2 under condition (II). Since we are assuming that $K_\mu < \infty$, in this case $\pm\infty$ are necessarily entrance boundaries. We have $|\{x \in \mathbb{R} : \mu(x) = \mu_{\text{inv}}(x)\}| = 2n$, for some $n \in \mathbb{N}$. The function $K_\mu(x), x \in \mathbb{R}$, possesses $k-1$ local minima and k local maxima, for some $k \in [n]$. Also $\pm\infty$ are local minima. If $k = 1$, then then the unique local maximum is the global maximum for K_μ, and the infimum is at $+\infty$, $-\infty$ or both.

Theorem 2 under condition (III-a). Since we are assuming that $K_\mu < \infty$, in this case necessarily $-\infty$ is an entrance boundary. We have $|\{x \in \mathbb{R} : \mu(x) = \mu_{\text{inv}}(x)\}| = 2n-1$, for some $n \in \mathbb{N}$. The function $K_\mu(x)$ possesses k local minima and k local maxima, for some $k \in \{0, \cdots, n-1\}$. Also, $-\infty$ is a local minimum and $+\infty$ is a local maximum. In particular, if $+\infty$ is not an entrance boundary, then $K_\mu(+\infty) = \infty$. If $k = 0$, then the infimum of K_μ is at $-\infty$ and the supremum of K_μ is at $+\infty$.

Theorem 2 under condition (III-b). The description above for condition (III-a) holds with the roles of $+\infty$ and $-\infty$ interchanged.

We now consider three concrete examples. For the diffusion process, we assume that the diffusion coefficient is $a(x) = 1$ and the drift is $b(x) = -\text{sgn}(x)\beta|x|^l$, with $l > -1$ and $\beta > 0$. By (1.4), the diffusion is positive recurrent; the boundaries $\pm\infty$ are entrance boundaries if and only if $l > 1$. Define

$$N(c,m) = \left(\int_{-\infty}^{\infty} e^{-c|w|^m} dw\right)^{-1}, \ c,m > 0. \tag{3.1}$$

By (1.3), the invariant density is given by

$$\mu_{\text{inv}}(x) = N(\frac{2\beta}{l+1}, l+1)e^{-\frac{2\beta}{l+1}|x|^{l+1}}, \ x \in \mathbb{R}. \tag{3.2}$$

We have the following proposition.

Proposition 1. *Let $a(x) = 1$ and $b(x) = -sgn(x)\beta|x|^l$, with $l > -1$ and $\beta > 0$. Then for all $x \in \mathbb{R}$,*

$$E_x\tau_y \sim \frac{1}{\beta N(\frac{2\beta}{l+1}, l+1)}|y|^{-l}e^{\frac{2\beta}{l+1}|y|^{l+1}}, \ \text{as } |y| \to \infty. \tag{3.3}$$

Remark. Note from (3.2) and (3.3) that K_μ is finite if and only if $l > 1$, and by (1.4), $\pm\infty$ are entrance boundaries if and only if $l > 1$, verifying in this particular case what the general result from [7] guarantees.

Proof of Proposition 1. By symmetry, it suffices to consider the case $y \to +\infty$. From (2.1), it is clear that with $a(x)$ and $b(x)$ as in the statement of the proposition, $\lim_{y\to+\infty} E_x\tau_y = \infty$, and consequently the leading order asymptotic behavior of $E_x\tau_y$ as $y \to +\infty$ is independent of x. Thus, we can assume that $x > 0$, in which case we have for $x < y$,

$$\begin{aligned} &E_x\tau_y = 2\int_x^y dz \exp(\frac{2\beta z^{l+1}}{l+1})\int_{-\infty}^z dw \exp(-\frac{2\beta|w|^{l+1}}{l+1}) \sim \\ &\frac{2}{N(\frac{2\beta}{l+1}, l+1)}\int_x^y \exp(\frac{2\beta z^{l+1}}{l+1})dz, \ \text{as } y \to +\infty. \end{aligned} \tag{3.4}$$

Using l'Hôpital's rule at ∞ shows that

$$\lim_{y\to+\infty}\frac{\int_x^y \exp(\frac{2\beta z^{l+1}}{l+1})dz}{(2\beta y^l)^{-1}\exp(\frac{2\beta y^{l+1}}{l+1})}=1. \tag{3.5}$$

From (3.4) and (3.5), we obtain (3.3) for $y\to+\infty$. □

For convenience, in what follows we will refer to the above diffusion process as a (β,l)-diffusion. Here now are three concrete examples.

1. *(β,l)-diffusion, with $l>-1$, and with target density* $\mu(x)=N(c,l+1)e^{-c|x|^{l+1}}$*, with* $c>\frac{2\beta}{l+1}$*:*

It follows from Proposition 1 that K_μ is finite. The diffusion and the target distribution correspond to condition (I) above. Clearly, $|\{x\in\mathbb{R}:\mu(x)=\mu_{\text{inv}}(x)\}|=2$. So by Theorem 2 applied to condition (I), we conclude that K_μ possesses a unique local minimum and that $\pm\infty$ are local maxima. By the symmetry, $\mu((-\infty,0])=\mu_{\text{inv}}((-\infty,0])=\frac{1}{2}$. Thus the unique local minimum of K_μ is at $x=0$, and this is the global minimum. By symmetry, the supremum occurs at both $\pm\infty$. In particular, if $l\in(-1,1]$, then $\pm\infty$ are not entrance boundaries, so $K_\mu(\pm\infty)=\infty$.

2. *(β,l) diffusion, with $l>-1$, and with target density* $\mu(x)=N(c,m)e^{-c|x|^m}$*, with $m>l+1$ and $c>0$:*

It follows from Proposition 1 that K_μ is finite. The diffusion and the target distribution correspond to condition (I) above. Thus, by Theorem 2 applied to condition (I), $\pm\infty$ are local maxima for K_μ. In particular, if $l\in(-1,1]$, then $\pm\infty$ are not entrance boundaries, so $K_\mu(\pm\infty)=\infty$. Clearly, there exists a $c_0=c_0(\beta)>0$ such that

$$\begin{aligned}&\mu(0)=N(c,m)<N(\frac{2\beta}{l+1},l+1)=\mu_{\text{inv}}(0),\quad \text{if } c<c_0;\\ &\mu(0)=N(c,m)>N(\frac{2\beta}{l+1},l+1)=\mu_{\text{inv}}(0),\quad \text{if } c>c_0;\end{aligned} \tag{3.6}$$

Thus, by symmetry and Theorem 2, K_μ has a maximum at $x=0$ if $c<c_0$ and a minimum at $x=0$ if $c>c_0$. For $c=c_0$, we have $\mu(0)=\mu_{\text{inv}}(0)$ and $\mu(x)-\mu_{\text{inv}}(x)$ is positive for $|x|$ sufficiently small, so by Theorem 2, K_μ has a minimum at $x=0$ if $c=c_0$. The equation $\mu(x)=\mu_{\text{inv}}(x)$, for $x>0$ is equivalent to $cx^m-\frac{2\beta}{l+1}x^{l+1}=\log\frac{N(c,m)}{N(\frac{2\beta}{l+1},l+1)}$, for $x>0$. The function $cx^m-\frac{2\beta}{l+1}x^{l+1}$ on $[0,\infty)$ vanishes at $x=0$ and it has a unique minimum, which is negative; it decreases to the left of that minimum and increases unboundedly to the right of that minimum. Note form (3.6) that $\log \frac{N(c,m)}{N(\frac{2\beta}{l+1},l+1)}$ is nonnegative if $c \ge c_0$ and negative if $c < c_0$. Thus, if $c \ge c_0$ it follows from the equation above that there is exactly one positive x for which $\mu(x) = \mu_{\text{inv}}(x)$. Consequently by Theorem 2 applied to condition (I) and by symmetery, K_μ has its unique local minimum at $x = 0$, which is the global minimum, and its supremum occurs at $\pm\infty$. On the other hand, for $c < c_0$, it follows from the equation above that there are either zero or two positive values of x for which $\mu(x) = \mu_{\text{inv}}(x)$. However, since both 0 and $\pm\infty$ are local maxima, it follows that K_μ must have a local minimum on $(0,\infty)$. Consequently, by Theorem 2 there must be two positive values of x for which $\mu(x) = \mu_{\text{inv}}(x)$. We conclude from Theorem 2 that K_μ has local maxima at 0 and $\pm\infty$, and two local minimum, which are the global minimum, at $\pm x_0$, where x_0 is the unique positive number which satisfies $\mu((-\infty,x_0]) = \mu_{\text{inv}}((-\infty,x_0])$.

3. *(β,l) diffusion, with $l > 1$, and with target density $\mu(x) = N_{a,\beta,l}\min(1,|x|^a)e^{-\frac{2\beta}{l+1}|x|^{l+1}}$, where $a \in (0,l-1)$ and $N_{a,\beta,l}$ is the appropriate normalization constant*:

It follows from Proposition 1 that K_μ is finite. The diffusion and the target distribution correspond to condition (II) above. By Theorem 2 applied to condition (II), $\pm\infty$ are local minima. Clearly, there is a unique positive x for which $\mu(x) = \mu_{\text{inv}}(x)$. Consequently, by symmetry and Theorem 2 applied to condition (II), we conclude that K_μ attains a unique local maximum, which is the global maximum, at $x = 0$ and its infimum occurs at $\pm\infty$.

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