A NEW LOOK AT SOME ASPECTS OF ONE-DIMENSIONAL RANDOM SEQUENTIAL ADSORPTION AND ITS CONTINUUM LIMIT

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Abstract. Fix a positive integer $k \geq 2$, and for $n \geq k$, consider a row of n molecules. From among the $n-k+1$ nearest-neighbor k-tuples of molecules, select one uniformly at random and bond the k molecules. Now, from all the remaining nearest-neighbor k-tuples, again select one uniformly at random and bond the k molecules. Continue like this until there are no nearest-neighbor k-tuples left. Let $M_k^{(n)}$ denote the expected value of the number of bonded molecules. An explicit integral formula for $m_k := \lim_{n\to\infty} \frac{M_k^{(n)}}{n}$ is known, and an explicit formula for $m_\infty := \lim_{k\to\infty} m_k$ is known. The constant m_∞, known as the Rényi parking constant, arises as the limiting packing density for a continuous analog of the above discrete packing problems. These are all models of what is called random sequential adsorption (RSA). The first part of this paper studies the gaps of sizes $0, 1, \cdots, k-1$ that arise between bonded k-tuples and shows that after scaling the k-grid, when $k \to \infty$ the empirical distribution of expected gaps in the discrete problem on the lattice converges weakly to an appropriate gap distribution that is known to hold for the above noted continuous analog. The second part of the this paper considers two different models of the discrete bonding problem when both k_1-bonding and k_2-bonding occur, with $2 \leq k_1 < k_2$. Explicit formulas are obtained for the analogs of m_k, and the asymptotic behavior of these analogs is studied both when $k_2 \to \infty$ with k_1 fixed, and when $k_1, k_2 \to \infty$ at certain ratios.

1. Introduction and Statement of Results

Fix a positive integer $k \geq 2$, and for $n \geq k$, consider a row of n molecules. From among the $n-k+1$ nearest-neighbor k-tuples of molecules, select one uniformly at random and bond the k molecules. Now, from all the

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remaining unbonded nearest-neighbor k-tuples, again select one uniformly at random and bond the k molecules. Continue like this until there are no unbonded nearest-neighbor k-tuples left. Let $M_k^{(n)}$ denote the expected value of the number of bonded molecules and consider $\lim_{n\to\infty}\frac{M_k^{(n)}}{n}$, the limiting expected density of bonded molecules as the number of molecules increases to infinity. The following theorem is known.

Theorem 1. *Let $M_k^{(n)}$ denote the expected number of bonded molecules under k-bonding on a row of n molecules. Then*

$$m_k := \lim_{n\to\infty}\frac{M_k^{(n)}}{n} = k\int_0^1 \exp(2\sum_{j=1}^{k-1}\frac{s^j-1}{j})\,ds. \tag{1.1}$$

The case $k=2$, proved by Flory [6] in 1939, thirty-five years before he won the Nobel Prize in Chemistry, appeared in a chemistry journal, and was then rediscovered and proved along with a corresponding weak law of large numbers by Page [12] in 1959 in a statistics journal. These proofs do not generalize to $k>2$. We proved the general case along with a corresponding weak law of large numbers in [13]. It turns out though that there were a number of earlier works that obtained the result (1.1), with varying levels of rigor and varying levels of detail; see for example, [9], [11], [7]. The integral in (1.1) can be calculated explicitly only when $k=2$; one has $m_2 = 1-e^{-2}\approx 0.865$. The values of m_k for various choices of k appear in the first column of table 2 below. It is clear from numerical evidence that m_k is decreasing in k; this was stated as a conjecture in [13], and seems to be open still.

One can show that

$$m_\infty := \lim_{k\to\infty} m_k = \int_0^\infty \exp\left(-2\int_0^x \frac{1-e^{-y}}{y}dy\right)dx \approx 0.7476. \tag{1.2}$$

The constant m_∞ is known as the Rényi parking constant. It arises directly from the following continuous packing problem. Consider the interval $[0,L]$. A car of length 1 wants to park on the interval. If $L<1$, then it can't park. If $L\geq 1$, the car chooses a point p uniformly at random from the interval $[0,L-1]$ and parks, taking up the interval $[p,p+1]$. This creates two smaller vacant intervals—$[0,p]$ and $[p+1,L]$. Now implement the above rule for each of these intervals. Continue like this until no more cars can park. Let $C^{(L)}$ denote the expected value of the length of space taken up by parked cars (or equivalently, of the number of parked cars). Rényi [14] proved that the limiting expected density of the space taken up by parked cars as the length of the interval increases to infinity is given by

$$\lim_{n\to\infty}\frac{C^{(L)}}{L}=m_\infty.$$

In a private communication [2], Julien Bureaux provided a proof of (1.2). We will reproduce it below at the end of this introductory section, as variants of it will be needed in this paper. In a literature search for this paper, we found an older similar proof of (1.2) in [4], which itself cites [9], where the result is demonstrated without complete mathematical rigor. If one scales the discrete molecule problem above by k, it is easy to see heuristically that the Rényi parking problem is the continuum limit of the discrete problem.

Both the discrete and the continuous models above are models of what is known in the chemistry literature as random sequential adsorption (RSA). For more about models of RSA and other related phenomena, the reader is referred to the review article [5]. From the mathematics literature point of view, the above problems are discrete and continuous random packing problems. For a fresh look at the discrete case with $k=2$, see [8]. For some refinements of Rényi's result, see for example, [3], [15] and [16].

This paper has two contributions. The first contribution is a study of the gaps between bonded k-tuples in the discrete model and a proof that after scaling the one-dimensional integer lattice by k, the empirical distribution of expected gaps in the discrete problem on the lattice converges weakly as $k\to\infty$ to an appropriate gap distribution that is known to hold for the Rényi parking problem. The second and more novel contribution is a study of two different models of the discrete bonding problem when both k_1-bonding and k_2-bonding occur, with $2\le k_1<k_2$. For the proofs of the results concerning one of these two models, the results on the gaps will play an essential role.

Turning to the first topic, we now describe the gaps. When k-bonding is completed on an interval of length n, there will be between any two consecutive bonded k-tuples a set of l unbonded molecules, for some $l\in\{0,1,\cdots,k-1\}$. Such an l-tuple of unbonded molecules also occurs before the leftmost bonded k-tuple and after the rightmost one. For $l\in$

$\{0,1,\cdots,k-1\}$, let $X^{(n)}_{k;l}$ denote the random variable counting the number of l gaps, and let $G^{(n)}_{k;l}=EX^{(n)}_{k;l}$ denote the expected value of the number of l-gaps. (We emphasize that this counts the number of l-gaps, not the number of molecules contained in the l-gaps, which is l times as large. Also, we are working with maximal gaps. That is, if there are l molecules between two consecutive k bonds, then this counts only as one l gap and not also as, say, two $l-1$ gaps.) A central limit theorem was proven for $X^{(n)}_{k;l}$ in [10]. That paper also proved the following theorem concerning $\lim_{n\to\infty}\frac{G^{(n)}_{k;l}}{n}$, the limiting expected number of l-gaps per unit length, that is, the limiting expected density of l-gaps, as the number of molecules increases to infinity. For completeness, we will provide a different but similar proof of the theorem, which is in the spirit of the proofs of the rest of the results in this paper, and which we proved before we were aware of [10].

Theorem 2. *Let $G^{(n)}_{k;l}$ denote the expected number of l gaps under k-bonding on a row of n molecules. Then*

(1.3)

$$g_{k;l}:=\lim_{n\to\infty}\frac{G^{(n)}_{k;l}}{n}=2\int_0^1(1-s)s^l\exp\left(2\sum_{j=1}^{k-1}\frac{s^j-1}{j}\right)ds,\ l=0,1,\cdots,k-1.$$

Remark. For $l=k-1$, the integral above can be calculated explicitly. One has $\left((1-s)^2\exp\left(2\sum_{j=1}^{k-1}\frac{s^j}{j}\right)\right)'=-2(1-s)s^{k-1}\exp\left(2\sum_{j=1}^{k-1}\frac{s^j}{j}\right)$, thus from (1.3), $g_{k;k-1}=\exp\left(-2\sum_{j=1}^{k-1}\frac{1}{j}\right)$.

Since $M^{(n)}_k$ is the expected number of molecules bonded in k-tuples, and since $\frac{M^{(n)}_k}{k}$ is the number of bonded k-tuples, it follows from the definitions that

$$\sum_{l=0}^{k-1}G^{(n)}_{k;l}=\frac{1}{k}M^{(n)}_k+1;$$
$$\sum_{l=1}^{k-1}lG^{(n)}_{k;l}+M^{(n)}_k=n,$$

and consequently by Theorem 2,

$$\begin{aligned}
&\sum_{l=0}^{k-1} g_{k;l} = \frac{m_k}{k};\\
&\sum_{l=1}^{k-1} l g_{k;l} = 1 - m_k.
\end{aligned} \tag{1.4}$$

We remind the reader that $g_{k;l}, lg_{k;l}, \frac{m_k}{k}$ and m_k are respectively the number of l-gaps, the number of molecules in l-gaps, the number of k-bonds and the number of molecules in k-bonds per unit length. We illustrate this in Table 1 for $k = 10$. (Due to roundoff error, there is a discrepancy in the fourth decimal place between the sum of the values in each of the two columns and the corresponding values appearing in the caption to the table.) It follows

l	$g_{10;l}$	$lg_{10;l}$
0	0.0187	0
1	0.0124	0.0124
2	0.0095	0.0190
3	0.0077	0.0232
4	0.0065	0.0260
5	0.0059	0.0280
6	0.0049	0.0293
7	0.0043	0.0303
8	0.0039	0.0310
9	0.0035	0.0314

TABLE 1. Gaps densities from Theorem 2 for $k = 10$; $\sum_{l=0}^{9} g_{10;l} = \frac{m_{10}}{10} \approx 0.0770$, $\sum_{l=1}^{9} lg_{10;l} = 1 - m_{10} \approx 0.2304$

from (1.3) that $g_{k;l}$ is decreasing for $l \in \{0, \cdots, k-1\}$. Numerical evidence suggests that $lg_{k;l}$ is increasing for $l \in \{0, \cdots, k-1\}$, but we don't have a proof.

In light of the first line of (1.4), define the scaled empirical measure of the expected gap densities for k-bonding by

$$\mu_k^{\mathrm{gaps}}([0,\gamma]) = \frac{k}{m_k} \sum_{l:\frac{l}{k-1} \le \gamma} g_{k;l}, \ \gamma \in [0,1]. \tag{1.5}$$

We will prove the following weak convergence result for the sequence $\{\mu_k^{\text{gaps}}\}_{k=2}^\infty$ of probability measures on $[0,1]$.

Theorem 3. *Consider the probability measures $\{\mu_k^{gaps}\}_{k=2}^\infty$ defined in (1.5), where $g_{k;l}$ is as in (1.3) and m_k is as in (1.1). Then*

(1.6)
$$\lim_{k\to\infty}\mu_k^{gaps}([0,\gamma])=\frac{2}{m_\infty}\int_0^\infty\left(1-e^{-\gamma x}\right)\exp\left(-2\int_0^x\frac{1-e^{-y}}{y}dy\right)dx,\ \gamma\in[0,1],$$

where m_∞ is the Rényi parking constant from (1.2). In particular, the limiting probability measure

$$\mu_\infty^{gaps}:=w-\lim_{k\to\infty}\mu_k^{gaps} \tag{1.7}$$

has density

$$f_{\mu_\infty^{gaps}}(\gamma)=\frac{2}{m_\infty}\int_0^\infty xe^{-\gamma x}\exp\left(-2\int_0^x\frac{1-e^{-y}}{y}dy\right)dx,\ \gamma\in[0,1]. \tag{1.8}$$

From (1.8), it is clear that $\lim_{\gamma\to0}f_{\mu_\infty^{\text{gaps}}}(\gamma)=\infty$. The graph of the density $f_{\mu_\infty^{\text{gaps}}}$ appears in figure 1. We have the following corollary.

Corollary 1. *The expected value of a μ_∞^{gaps}-distributed random variable is given by*

$$\int_0^1\gamma f_{\mu_\infty^{gaps}}(\gamma)d\gamma=\frac{1-m_\infty}{m_\infty}\approx0.3376. \tag{1.9}$$

Remark. The corollary indicates that for k-bonding with very large k, the average size of a gap is just slightly more than $\frac{1}{3}$ the maximum gap size of $k-1$.

Proof. Using (1.6) and (1.5) for the first equality below and the second line of (1.4) for the second equality below, we have

$$\int_0^1\gamma f_{\mu_\infty^{\text{gaps}}}(\gamma)d\gamma=\lim_{k\to\infty}\frac{k}{m_k}\sum_{l=1}^{k-1}\frac{l}{k-1}g_{k;l}=\frac{1-m_\infty}{m_\infty}.$$

□

In the Rényi parking problem on an interval of length L, gaps between parked cars are formed, with gap lengths in the interval $[0,1)$. Bánkövi [1] carefully constructed the appropriate probability space for this problem and then defined a gap chosen uniformly at random from all of the existing gaps. The distribution of this gap is of course a probability measure on

$[0,1]$. He showed that this gap distribution converges weakly as $L \to \infty$ to the probability measure μ_∞^{gaps} defined in (1.7). Thus, Theorem 3 shows that the appropriately scaled empirical distribution of expected gaps in the discrete problem on the lattice converges as $k \to \infty$ to the corresponding appropriate object for the Rényi problem on the line.

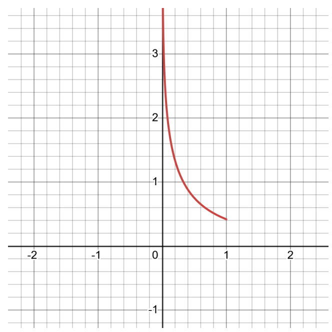

FIGURE 1. The density $f_{\mu_\infty^{\text{gaps}}}$ of μ_∞^{gaps} from Theorem 3.

We now turn to a study of the discrete bonding problem when both k_1-bonding and k_2-bonding occur, with $2 \le k_1 < k_2$. We will consider two different models. In the first model, k_2 bonding is performed to its completion on a row of n molecules. This leaves gaps of sizes between 0 and $k_2 - 1$. Now k_1-bonding is implemented on these gaps. Then as before, we let $n \to \infty$. In the second model, at first k_1 and k_2 bonding occur together. At each step, one considers all of the possible k_1-tuples and all of the possible k_2-tuples, chooses one of them uniformly at random and implements the bond. When there are no longer any unbonded k_2-tuples, k_1-bonding continues on all the remaining gaps, whose lengths are of course at most $k_2 - 1$. Then we let $n \to \infty$. Both models can be considered to have two stages. In model I, the first stage is k_2-bonding and the second stage is k_1-bonding on the remaining gaps of lengths less than k_2. In model II, the first stage is competitive k_1 and k_2-bonding, continuing until there are no longer any unbonded k_2-tuples. The second stage is k_1-bonding on the remaining gaps of lengths less than k_2. We will be interested in results for k_1 and k_2 fixed, for k_1 fixed and $k_2 \to \infty$, and for $k_1, k_2 \to \infty$ at certain relative rates. Below we display the definitions of the two models and define the notation for the expected values of the number of bonded molecules:

(1.10)

Model I: Let $2 \le k_1 < k_2 \le n$. Implement k_2-bonding on the line of n molecules, and when it is completed, implement k_1-bonding on the resulting gaps. Let $M^{(n)}_{k_1,k_2;I}$ denote the expected number of bonded molecules. Now let $n \to \infty$.

Model II: Let $2 \le k_1 < k_2 \le n$. Implement k_1 and k_2-bonding together on the line of n molecules—at each step choose uniformly at random from all of the possible k_1-tuples and k_2-tuples, and bind the molecules. When there are no longer any unbonded k_2-tuples, continue with k_1-bonding. Let $M^{(n)}_{k_1,k_2;II}$ denote the expected number of bonded molecules, and let $M^{(n)}_{k_1,k_2;II_1}$ denote the expected number of bonded molecules at the end of stage one, when there are no longer any unbonded k_2-tuples. Now let $n \to \infty$.

Remark. Note that the parallel of $M^{(n)}_{k_1,k_2;II_1}$ for model I is $M^{(n)}_{k_2}$, the expected number of bonded molecules for k_2-bonding on a row of n molecules.

We now consider $\lim_{n\to\infty} \frac{M^{(n)}_{k_1,k_2;I}}{n}$ and $\lim_{n\to\infty} \frac{M^{(n)}_{k_1,k_2;II}}{n}$, the limiting expected density of bonded molecules as the number of molecules increases to infinity in the two models, and also $\lim_{n\to\infty} \frac{M^{(n)}_{k_1,k_2;II_1}}{n}$, the limiting expected density of bonded molecules at the end of stage one in model II, as the number of molecules increases to infinity. Of course, the limiting expected density of bonded molecules at the end of stage one in model I as the number of molecules increases to infinity is $\lim_{n\to\infty} \frac{M^{(n)}_{k_2}}{n} = m_{k_2}$, which appears in Theorem 1.

For Model I, we have the following theorem, whose proof is immediate in light of Theorems 1 and 2.

Theorem 4. *Let $M^{(n)}_{k_1,k_2;I}$ denote the expected number of bonded molecules under Model I in* (1.10). *Then*

$$m_{k_1,k_2;I} := \lim_{n\to\infty} \frac{M^{(n)}_{k_1,k_2;I}}{n} = m_{k_2} + \sum_{l=k_1}^{k_2-1} g_{k_2;l} M^{(l)}_{k_1}, \tag{1.11}$$

where m_{k_2} is as in Theorem 1, $g_{k_2;l}$ is as in Theorem 2 and $M_{k_1}^{(l)}$ is the expected number of bonded molecules under k_1-bonding on a row of l molecules.

Proof. In Model 1, first k_2-bonding is implemented. When it is completed, between any two consecutive k-tuples, there will be a gap of size of between 0 and k_2-1. On these gaps, k_1-bonding is implemented. The proof follows from this description along with Theorems 1 and 2 and the linearity of the expectation. □

We will prove the following theorem for Model II.

Theorem 5. *Let $M_{k_1,k_2;II}^{(n)}$ denote the expected number of bonded molecules under Model II in (1.10), and let $M_{k_1,k_2;II_1}^{(n)}$ denote the expected number of bonded molecules at the end of stage one, when there are no longer any unbonded k_2-tuples. Then*

(1.12)

$$m_{k_1,k_2;II_1} := \lim_{n\to\infty} \frac{M_{k_1,k_2;II_1}^{(n)}}{n} =$$
$$\frac{1}{2}\int_0^1 s^{\frac{k_2-k_1}{2}}\big(k_1+k_2+k_1(k_2-k_1)(1-s)\big)\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)ds,$$

and

(1.13)

$$m_{k_1,k_2;II} := \lim_{n\to\infty}\frac{M_{k_1,k_2;II}^{(n)}}{n} = m_{k_1,k_2;II_1} + \int_0^1 B(s)\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)ds,$$

where

(1.14)

$$B(s) = (1-s)^2 s^{-\frac{1}{2}k_1-\frac{1}{2}k_2}\left(\sum_{n=\max(k_2,2k_1)}^{k_2+k_1-1}\Big(\sum_{j=k_1}^{n-k_1} M_{k_1}^{(j)}\Big)s^n + \sum_{n=k_2+k_1}^{2k_2-1}\Big(\sum_{j=k_1}^{n-k_2} M_{k_1}^{(j)}\Big)s^n\right)+$$
$$(1-s)\left(s^{\frac{1}{2}k_2+\frac{1}{2}k_1}+s^{\frac{3}{2}k_2-\frac{1}{2}k_1}\right)\left(\sum_{j=k_1}^{k_2-1}M_{k_1}^{(j)}\right),$$

and $M_{k_1}^{(j)}$ is the expected number of bonded molecules under k_1-bonding on a row of j molecules.

Remark. The explicit integral expressions obtained in Theorems 4 for $m_{k_1,k_2;I}$ and in Theorem 5 for $m_{k_1,k_2;II}$ depend on the values of $M_{k_1}^{(j)}$, for

$j \in \{k_1, \cdots, k_2 - 1\}$. (In Theorem 4 the integral expressions arise through the terms $g_{k_2;l}$.) For small k_2, these values can be figured out directly. In the case of general k_2, if $k_2 \le 2k_1$, then from the definition of the k_1-bonding model, all of these values are immediate: $M_{k_1}^{(j)} = k_1,\ j = k_1, \cdots, k_2 - 1$.

Table 2 compares the values $m_{k_1}, m_{k_1,k_2,I}$ and $m_{k_1,k_2,II}$ for various values of k_1, with $k_2 = k_1 + 1$ and with $k_2 = 2k_1$. We conjecture that $m_{k_1;k_2;I}$ and $m_{k_1,k_2;II}$ are increasing in k_2 for $k_2 > k_1 \ge 2$.

k_1	m_{k_1}	$m_{k_1,k_1+1;I}$	$m_{k_1,2k_1;I}$	$m_{k_1,k_1+1;II}$	$m_{k_1,2k_1;II}$
2	0.865	0.923	0.924	0.892	0.894
3	0.824	0.881	0.900	0.850	0.860
5	0.793	0.837	0.880	0.813	0.833
10	0.770	0.796	0.866	0.782	0.814
100	0.750	0.753	0.854	0.751	0.796

Table 2. Limiting bonding densities under various regimes from Theorems 1, 4 and 5.

In the remark after Theorem 5, it was seen that one can calculate generically $M_{k_1}^{(j)}$ essentially only when $j \le 2k_1$. (We say "essentially" because actually it is easy to figure out by hand $M_{k_1}^{(j)}$ for $j = 2k_1 + 1$, and then progressively harder for $j = 2k_1 + 2, \cdots$.) But Theorem 1 gives the asymptotic behavior of $M_{k_1}^{(j)}$ as $j \to \infty$. In the formulas for $m_{k_1,k_2;I}$ and $m_{k_1,k_2;II}$, the influence of $M_{k_1}^{(j)}$ for any particular j becomes negligible as $k_2 \to \infty$. (This can be seen easily in Theorem 4 since $\lim_{k_2\to\infty} g_{k_2;l} = 0$, for any l. That this limit is zero can be seen by considering (3.1) with the term $\sum_{l=0}^{L} s^l$ replaced by s^l.) This allows us to evaluate the limits $\lim_{k_2\to\infty} m_{k_1,k_2;I}$ and $\lim_{k_2\to\infty} m_{k_1,k_2;II}$ for any fixed k_1. We can also evaluate the limits $\lim_{k_1\to\infty} m_{k_1,Lk_1;I}$ and $\lim_{k_1\to\infty} m_{k_1,Lk_1;II}$ with $L \in (1,2]$. The reason that $k_2 = Lk_1$ is restricted to $L \in (1,2]$ is the issue noted in the first line of this paragraph. We begin with the results for fixed k_1.

Theorem 6. *Let $m_{k_1;k_2;I}$ be as in Theorem 4. One has*

(1.15)

$$m_{k_1,\infty;I} := \lim_{k_2\to\infty} m_{k_1,k_2;I} = m_\infty + (1 - m_\infty) m_{k_1} \approx 0.7476 + 0.252\, m_{k_1}.$$

Thus also,

$$\lim_{k_1\to\infty} m_{k_1,\infty;I} = m_\infty + (1-m_\infty)m_\infty \approx 0.9363.$$

Remark. Theorem 6 has a quick proof using Theorems 1, 2 and 4, but we don't present it here in order not to interrupt the exposition. Theorem 6 is easy to explain intuitively. Fix a very large k_2 and let n, the total number of molecules in the row, be much larger than k_2. In Model I, first k_2-bonding is implemented. Since n is much larger than k_2, by Theorem 1 the percentage of bonded molecules is close to m_{k_2}, and since k_2 is very large, m_{k_2} is close to m_∞. This accounts for the term m_∞ in (1.15). Now k_1-bonding is implemented on the gaps. The percentage of the original n molecules that are in these gaps is close to $1-m_{k_2}$, which in turn is close to $1-m_\infty$. Since the limiting density of gaps $g_{k_2;l}$ as $n\to\infty$ satisfy $\lim_{k_2\to\infty} g_{k_2;l}=0$, for any fixed l, it follows that for our very large n, after the k_2-bonding is implemented, most of the gaps on which k_1-bonding is now implemented are large. Thus, by Theorem 1, the percentage of molecules that were unbonded after the k_2-bonding was completed and which now get bonded during the k_1-bonding phase is close to m_{k_1}. This accounts for the term $m_{k_1}(1-m_\infty)$ in (1.15).

Theorem 7. *Let $m_{k_1;k_2;II}$ and $m_{k_1;k_2;II_1}$ be as in Theorem 5. One has*

$$m_{k_1,\infty;II_1} := \lim_{k_2\to\infty} m_{k_1,k_2;II_1} = D \approx 0.4166, \tag{1.16}$$

and

$$m_{k_1,\infty;II} := \lim_{k_2\to\infty} m_{k_1,k_2;II} = D+(1-D)m_{k_1} \approx 0.4166+0.5834\, m_{k_1}, \tag{1.17}$$

where

$$D=\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}x}e^{-\int_0^x \frac{1-e^{-y}}{y}dy}dx \approx 0.4166. \tag{1.18}$$

Thus also,

$$\lim_{k_1\to\infty} m_{k_1,\infty;II} = D+(1-D)m_\infty \approx 0.8528.$$

Remark. Analogous to Theorem 6 and the remark following that theorem, Theorem 7 has the following explanation. In Model II, k_1-bonding and k_2-bonding take place in a competing manner until k_2 bonding is no longer possible. Then k_1-bonding continues alone until it is completed. Equation

(1.16) shows that for fixed k_1 and very large k_2, at the point that k_2-bonding is no longer possible, the percentage of bonded molecules (both k_1-bonds and k_2-bonds) is approximately D. After this, k_1 bonding continues to be implemented on the remaining molecules whose percentage is approximately $1-D$.

We note two very interesting points concerning the above remark. First of all, the value of D is independent of k_1, despite the fact that in Model II, unlike in Model 1 analyzed in Theorem 6, k_1 and k_2 bonding are competing at the same time. Second of all, *the value of D (≈ 0.4166) is drastically lower than the value of m_∞ (≈ 0.7476). This shows that the k_1-sized bonds that occur during the competing stage of k_1 and k_2 bonding heavily reduce the number of consecutive unbonded k_2-tuples available for k_2-bonding.* This carries over heuristically to the continuous parking model of Renyi. Let k_2 be very large. Scaling by k_2, we think of the k_2 bond as being a clump of bonded molecules of unit length (corresponding to the parked car of length 1). The bonds of length k_1, when scaled by k_2, now represent minute particles that interfere with bonding.

Table 3 compares the values of $m_{k_1}, m_{k_1,\infty;I}$ and $m_{k_1,\infty;II}$ for various values of k_1. One might also want to compare Table 3 with Table 2. Tables 2 and 3 show that for fixed k_1, and any choice of k_2, or alternatively $k_2 \to \infty$, the highest percentage of bonded molecules occurs for Model I k_1, k_2-bonding, then for Model II k_1, k_2-bonding, and then the lowest percentage is for the standard k_1-bonding.

k_1	m_{k_1}	$m_{k_1,\infty;I}$	$m_{k_1,\infty;II}$
2	0.865	0.966	0.921
3	0.824	0.956	0.897
5	0.792	0.948	0.879
10	0.770	0.942	0.866
100	0.750	0.937	0.854

TABLE 3. Limiting bonding densities under k_1-bonding and under Model I and Model II k_1, k_2-bonding when $k_2 \to \infty$, from Theorems 1, 6 and 7.

We now evaluate $\lim_{k_1\to\infty} m_{k_1,Lk_1;I}$ and $\lim_{k_1\to\infty} m_{k_1,Lk_1;II}$ with $L \in (1,2]$.

Theorem 8. *Let $m_{k_1,k_2;I}$ be as in Theorem 4. Then for $L \in (1,2]$,*

$$m_{\infty,L\infty;I} := \lim_{k_1\to\infty} m_{k_1,[Lk_1];I} = m_\infty + \frac{2}{L}\int_0^\infty \left(e^{-\frac{1}{L}x} - e^{-x}\right) e^{-2\int_0^x \frac{1-e^{-y}}{y}dy}dx. \tag{1.19}$$

Remark. Note that if one substitutes $L = 1$ in (1.19), the expression reduces to m_∞ as is to be expected.

As with Theorem 6, Theorem 8 has a rather quick proof using Theorems 1, 2 and 4, but again we don't present it here in order to not interrupt the exposition .

Theorem 9. *Let $m_{k_1,k_2;II}$ be as in Theorem 5. Then for $L \in (1,2]$,*

$$\begin{aligned} &m_{\infty,L\infty;II} := \lim_{k_1\to\infty} m_{k_1,[Lk_1];II} = \\ &\frac{1}{2}\int_0^\infty (L+1+(L-1)x)\, e^{-\frac{L-1}{2}x} e^{-\int_0^x \frac{1-e^{-y}}{y}dy} e^{-\int_0^{Lx} \frac{1-e^{-y}}{y}dy}dx + \\ &\int_0^\infty \left(e^{-\frac{3-L}{2}x} - e^{-\frac{3L-1}{2}x}\right) e^{-\int_0^x \frac{1-e^{-y}}{y}dy} e^{-\int_0^{Lx} \frac{1-e^{-y}}{y}dy}dx. \end{aligned} \tag{1.20}$$

Remark. Note that if one substitutes $L = 1$ in (1.20), the expression reduces to m_∞ in (1.2), as is to be expected.

Table 4 compares the values of $m_{\infty,L\infty;I}$ and $m_{\infty,L\infty;II}$ for various values of L.

L	$m_{\infty,L\infty;I}$	$m_{\infty,L\infty;II}$
1.2	0.795	0.768
1.4	0.822	0.779
1.6	0.838	0.785
1.8	0.847	0.790
2.0	0.852	0.794

TABLE 4. Limiting bonding densities under Model I and Model II k_1, k_2-bonding with $k_2 = Lk_1$ and $k_1 \to \infty$, from Theorems 8 and 9.

It would be interesting to understand the behavior of $\lim_{k_1\to\infty} m_{k_1,Lk_1;I}$ and $\lim_{k_1\to\infty} m_{k_1,Lk_1;II}$ also for the case $L>2$ as well as for the case that $L=L(k_1)\to\infty$ at various rates.

As noted earlier, we end the introduction with the proof of (1.2).

Proof of (1.2). Making the substitution $y=k(1-t)$ in the third equality below, we write

(1.21)
$$\sum_{j=1}^{k-1}\frac{s^j-1}{j}=-\sum_{j=1}^{k-1}\int_s^1 t^{j-1}dt=-\int_s^1\frac{1-t^{k-1}}{1-t}dt=-\int_0^{k(1-s)}\frac{1-(1-\frac{y}{k})^{k-1}}{y}dy.$$

Therefore, making the substitution $x=k(1-s)$ in the third equality below, we have

(1.22)
$$k\int_0^1 e^{2\sum_{j=1}^{k-1}\frac{s^j-1}{j}}ds=k\int_0^1\exp\left(-2\int_0^{k(1-s)}\frac{1-(1-\frac{y}{k})^{k-1}}{y}dy\right)ds=$$
$$\int_0^k\exp\left(-2\int_0^x\frac{1-(1-\frac{y}{k})^{k-1}}{y}dy\right)dx\stackrel{k\to\infty}{\to}\int_0^\infty\exp\left(-2\int_0^x\frac{1-e^{-y}}{y}dy\right)dx,$$

where the limit follows by applying the dominated convergence theorem. This proves (1.2). □

We prove Theorem 2 in section 2, Theorem 3 in section 3, and Theorems 5-9 respectively in sections 4-8.

2. Proof of Theorem 2

In [13], we proved Theorem 1 by analyzing the generating function of the sequence $\{S_k^{(n)}\}_{n=1}^\infty$, where $S_k^{(n)}=\sum_{i=1}^n M_k^{(i)}$. The reason we chose to work with the generating function of $\{S_k^{(n)}\}_{n=1}^\infty$ instead of with the generating function of $\{M_k^{(n)}\}_{n=1}^\infty$ is that the former sequence satisfies a three term recurrence formula while the latter sequence satisfies a recursion formula that grows with n. This was good enough for the proof of Theorem 1, but it does not allow for the proof of Theorem 2. The method we employ here can be slightly modified to prove Theorem 1, as we point out below after completing the proof of Theorem 2.

As noted in the introduction, after k-bonding on a row of n molecules is completed, between every two consecutive bonded k-tuples there will be a gap of l unbonded molecules, for some $l\in\{0,1,\cdots,k-1\}$. There will also be such gaps before the leftmost bonded k-tuple and after the rightmost bonded k-tuple. For $n \geq 1$, let $G_{k;l}^{(n)}$ denote the expected number of such gaps of length l. Define $G_{k;l}^{(0)} = 0$, if $l \neq 0$ and $G_{k;0}^{(0)} = 1$. From the definition of k bonding on a row of n molecules, we have

$$G_{k;l}^{(n)} = \frac{1}{n-k+1}\sum_{j=1}^{n-k+1}\left(G_{k;l}^{(j-1)} + G_{k;l}^{(n+1-j-k)}\right) = \frac{2}{n-k+1}\sum_{j=0}^{n-k} G_{k;l}^{(j)},\ n \geq k, \tag{2.1}$$

with the boundary condition

$$G_{k;l}^{(n)} = \begin{cases} 1, n = l; \\ 0, n \in \{0, \cdots, k-1\} - \{l\}. \end{cases} \tag{2.2}$$

Multiplying both sides of (2.1) by $(n-k+1)t^n$ and summing over n from k to ∞, we obtain

$$\sum_{n=k}^{\infty}(n-k+1)G_{k;l}^{(n)}t^n = 2\sum_{n=k}^{\infty}\left(\sum_{j=0}^{n-k} G_{k;l}^{(j)}\right)t^n. \tag{2.3}$$

Define the generating function

$$g_{k;l}(t) = \sum_{n=k}^{\infty} G_{k;l}^{(n)} t^n. \tag{2.4}$$

Using the boundary condition (2.2), we write the right hand side of (2.3) as

$$\sum_{n=k}^{\infty}\left(\sum_{j=0}^{n-k} G_{k;l}^{(j)}\right)t^n = \sum_{n=k}^{2k-1}\left(\sum_{j=0}^{n-k} G_{k;l}^{(j)}\right)t^n + \sum_{n=2k}^{\infty}\left(\sum_{j=0}^{n-k} G_{k;l}^{(j)}\right)t^n = \sum_{n=k+l}^{2k-1} t^n + \sum_{n=2k}^{\infty}\left(1 + \sum_{j=k}^{n-k} G_{k;l}^{(j)}\right)t^n = \sum_{n=k+1}^{\infty} t^n + \sum_{n=2k}^{\infty}\left(\sum_{j=k}^{n-k} G_{k;l}^{(j)}\right)t^n. \tag{2.5}$$

We have

$$t^k\,\frac{g_{k;l}(t)}{1-t} = t^k\left(\sum_{r=0}^{\infty} t^r\right)\left(\sum_{j=k}^{\infty} G_{k;l}^{(j)}t^j\right) = \sum_{n=2k}^{\infty}\left(\sum_{j=k}^{n-k} G_{k;l}^{(j)}\right)t^n. \tag{2.6}$$

From (2.3)-(2.6), we conclude that

$$tg_{k;l}'(t) - (k-1)g_{k;l}(t) = \frac{2t^{k+l}}{1-t} + \frac{2t^k g_{k;l}(t)}{1-t},$$

which we write as

$$g'_{k;l}(t) = A(t)g_{k;l}(t) + B(t), \text{ where}$$
$$A(t) = \frac{k-1}{t} + \frac{2t^{k-1}}{1-t}; \tag{2.7}$$
$$B(t) = \frac{2t^{k+l-1}}{1-t},$$

the dependence of A on k and of B on k and l being suppressed. Solving the linear ODE gives for any $\epsilon \in (0,1)$,

$$g_{k;l}(t) = g_{k;l}(\epsilon)e^{\int_\epsilon^t A(s)ds} + \int_\epsilon^t e^{\int_s^t A(r)dr}B(s)ds, \ \epsilon < t < 1. \tag{2.8}$$

From (2.7), we write

$$A(t) = \frac{k-1}{t} + \frac{2}{1-t} - 2\sum_{i=0}^{k-2} t^i,$$

from which it follows that

$$e^{\int_s^t A(r)dr} = \frac{t^{k-1}}{s^{k-1}}\frac{(1-s)^2}{(1-t)^2}e^{2\sum_{j=1}^{k-1}\frac{s^j-t^j}{j}}. \tag{2.9}$$

By (2.4), $g_{k;l}(\epsilon) = O(\epsilon^k)$, thus it follows from (2.9) that $\lim_{\epsilon\to 0} g(\epsilon)e^{\int_\epsilon^t A(s)ds} = 0$. Thus, from (2.7)-(2.9) we conclude that

$$g_{k;l}(t) = \frac{2t^{k-1}}{(1-t)^2}\int_0^t (1-s)s^l e^{2\sum_{j=1}^{k-1}\frac{s^j-t^j}{j}}ds. \tag{2.10}$$

It is easy to see that if $\lim_{n\to\infty}\frac{G^{(n)}_{k;l}}{n} = a$, then $\lim_{t\to 1^-}(1-t)^2 g_{k;l}(t) = a$. Indeed, for any $\epsilon > 0$, there exists an N_ϵ such that $a - \epsilon \le \frac{G^{(n)}_{k;l}}{n} \le a + \epsilon$, for $n \ge N\epsilon$. Write

$$\sum_{n=k}^{N_\epsilon-1} G^{(n)}_{k;l}t^n + \sum_{n=N_\epsilon}^{\infty}(a-\epsilon)nt^n \le g_{k;l}(t) \le \sum_{n=k}^{N_\epsilon-1} G^{(n)}_{k;l}t^n + \sum_{n=N_\epsilon}^{\infty}(a+\epsilon)nt^n. \tag{2.11}$$

One has

$$\sum_{n=N_\epsilon}^{\infty} bnt^n = bt\left(\frac{t^{N_\epsilon}}{1-t}\right)' = bt\left(\frac{t^{N_\epsilon}}{(1-t)^2} + \frac{N_\epsilon t^{N_\epsilon-1}}{1-t}\right), \text{ for } b > 0. \tag{2.12}$$

From (2.11) and (2.12), it follows that

$$a - \epsilon \le \liminf_{t\to 1^-}(1-t)^2 g_{k;l}(t) \le \limsup_{t\to 1^-}(1-t)^2 g_{k;l}(t) \le a + \epsilon.$$

Unfortunately, we don't see any very quick way to prove that $\lim_{n\to\infty}\frac{G^{(n)}_{k;l}}{n}$ exists. Define $S^{(n)}_{k;l}=\sum_{j=0}^{n}G^{(j)}_{k;l}$. From (2.1), one has

$$G^{(n)}_{k;l}=\frac{2}{n-k+1}S^{(n-k)}_{k;l}.$$

Thus, $\lim_{n\to\infty}\frac{G^{(n)}_{k;l}}{n}$ exists if and only if $\lim_{n\to\infty}\frac{S^{(n)}_{k;l}}{n^2}$ exists. The proof that this latter limit exists is almost exactly the same as the proof of Proposition 3.3 in [13]. In light of this, we conclude from (2.10) that

$$\lim_{n\to\infty}\frac{G^{(n)}_{k;l}}{n}=\lim_{t\to1^-}(1-t)^2g_{k;l}(t)=2\int_0^1(1-s)s^le^{2\sum_{j=1}^{k-1}\frac{s^j-1}{j}}ds,$$

which proves Theorem 2. □

We note that the method above can also be use to prove Theorem 1. Instead of (2.1), one has

$$M^{(n)}_k=\frac{1}{n-k+1}\sum_{j=1}^{n-k+1}\left(M^{(j-1)}_k+M^{(n+1-j-k)}_k+k\right)=$$
$$k+\frac{2}{n-k+1}\sum_{j=0}^{n-k}M^{(j)}_{k;l},\ n\ge k,$$

with boundary condition $M^{(n)}_k=0$, for $n=0,\cdots k-1$.

3. Proof of Theorem 3

Making the same substitutions used in (1.21) and (1.22), and recalling (1.3), we have

(3.1)
$$\sum_{l=0}^{L}g_{k;l}=2\int_0^1(1-s)\left(\sum_{l=0}^{L}s^l\right)e^{2\sum_{j=1}^{k-1}\frac{s^j-1}{j}}ds=2\int_0^1(1-s^{L+1})e^{2\sum_{j=1}^{k-1}\frac{s^j-1}{j}}ds=$$
$$\frac{2}{k}\int_0^k\left(1-(1-\frac{x}{k})^{L+1}\right)\exp\left(-2\int_0^x\frac{1-(1-\frac{y}{k})^{k-1}}{y}dy\right)dx,\ L=0,1,2,\cdots.$$

Recalling the definition of μ_k^{gaps} in (1.5), and using (3.1) for the final equality below, we have

(3.2)

$$\mu_k^{\text{gaps}}([0,\gamma]) = \frac{k}{m_k}\sum_{l:\frac{l}{k-1}\le\gamma} g_{k;l} = \frac{k}{m_k}\sum_{l=0}^{[\gamma(k-1)]} g_{k;l} =$$

$$\frac{2}{m_k}\int_0^k \Big(1-(1-\frac{x}{k})^{[\gamma(k-1)]+1}\Big)\exp\left(-2\int_0^x \frac{1-(1-\frac{y}{k})^{k-1}}{y}dy\right)dx, \text{for } \gamma\in(0,1].$$

Theorem 3 now follows from (1.2) and from applying the dominated convergence theorem upon letting $k\to\infty$ in (3.2). □

4. Proof of Theorem 5

From the definition of Model II k_1,k_2-bonding, it follows that $M^{(n)}_{k_1,k_2;II}$, the expected value of the number of bonded molecules from a row of n molecules, satisfies

(4.1)

$$M^{(n)}_{k_1,k_2;II} = \frac{1}{(n-k_1+1)+(n-k_2+1)}\times$$

$$\left(\sum_{j=1}^{n-k_1+1}\Big(M^{(j-1)}_{k_1,k_2;II}+M^{(n-j-k_1+1)}_{k_1,k_2;II}+k_1\Big)+\sum_{j=1}^{n-k_2+1}\Big(M^{(j-1)}_{k_1,k_2;II}+M^{(n-j-k_2+1)}_{k_1,k_2;II}+k_2\Big)\right),$$

$n\ge k_2$,

with the boundary condition

$$M^{(n)}_{k_1,k_2;II} = \begin{cases} M^{(n)}_{k_1},\ n=k_1,\cdots,k_2-1;\\ 0,\ l=0,\cdots,k_1-1,\end{cases} \tag{4.2}$$

where we recall that $M^{(n)}_{k_1}$ denotes the expected number of bonded molecules for a row of n molecules under k_1-bonding. Also, from the definition of Model II k_1,k_2-bonding, it follows that $M^{(n)}_{k_1,k_2;II_1}$, the expected value of the number of bonded molecules from a row of n molecules at the end of stage one when there are no longer any unbonded k_2-tuples, satisfies the very same equation (4.1), but this time with boundary condition

$$M^{(n)}_{k_1,k_2;II_1}=0,\ l=0,\cdots,k_2-1.$$

Thus, in light of the linearity of equation (4.1), it suffices to prove only (1.13) and (1.14), where in (1.13) we understand the term $m_{k_1,k_2;II_1}$ to be the right hand side of (1.12).

Multiplying both sides of (4.1) by $(2n-k_1-k_2+2)t^n$, and summing over n from k_2 to ∞, we obtain

(4.3)
$$\sum_{n=k_2}^{\infty}(2n-k_1-k_2+2)M^{(n)}_{k_1,k_2;II}t^n = \sum_{n=k_2}^{\infty}\left(k_1(n-k_1+1)+k_2(n-k_2+1)\right)t^n+$$
$$2\sum_{n=k_2}^{\infty}\left(\sum_{j=0}^{n-k_1}M^{(j)}_{k_1,k_2;II}\right)t^n + 2\sum_{n=k_2}^{\infty}\left(\sum_{j=0}^{n-k_2}M^{(j)}_{k_1,k_2;II}\right)t^n.$$

Define the generating function

$$m_{k_1,k_2}(t) = \sum_{n=k_2}^{\infty}M^{(n)}_{k_1,k_2;II}t^n. \tag{4.4}$$

Consider the first term on the right hand side of (4.3). A direct calculation gives

$$\begin{aligned}\sum_{n=k_2}^{\infty}(n-k_1+1)t^n &= \frac{\big[(k_2-k_1+1)-(k_2-k_1)t\big]t^{k_2}}{(1-t)^2};\\ \sum_{n=k_2}^{\infty}(n-k_2+1)t^n &= \frac{t^{k_2}}{(1-t)^2}.\end{aligned} \tag{4.5}$$

Now consider the second term on the right hand side of (4.3). Using the boundary condition (4.2), similar to (2.5) we have

(4.6)
$$\sum_{n=k_2}^{\infty}\left(\sum_{j=0}^{n-k_1}M^{(j)}_{k_1,k_2;II}\right)t^n = \sum_{n=\max(k_2,2k_1)}^{k_2+k_1-1}\left(\sum_{j=k_1}^{n-k_1}M^{(j)}_{k_1}\right)t^n + \sum_{n=k_2+k_1}^{\infty}\left(\sum_{j=k_1}^{k_2-1}M^{(j)}_{k_1}\right)t^n+$$
$$\sum_{n=k_2+k_1}^{\infty}\left(\sum_{j=k_2}^{n-k_1}M^{(j)}_{k_1,k_2;II}\right)t^n.$$

Note that the second term on the right hand side of (4.6) can be written as

$$\sum_{n=k_2+k_1}^{\infty}\left(\sum_{j=k_1}^{k_2-1}M^{(j)}_{k_1}\right)t^n = \left(\sum_{j=k_1}^{k_2-1}M^{(j)}_{k_1}\right)\frac{t^{k_2+k_1}}{1-t}. \tag{4.7}$$

The final term on the right hand side of (4.6) can be dealt with similar to (2.6). We have
(4.8)

$$t^{k_1}\,\frac{m_{k_1,k_2}(t)}{1-t}=t^{k_1}\left(\sum_{r=0}^\infty t^r\right)\left(\sum_{j=k_2}^\infty M^{(j)}_{k_1,k_2;II}t^j\right)=\sum_{n=k_2+k_1}^\infty\left(\sum_{j=k_2}^{n-k_1}M^{(j)}_{k_1,k_2;II}\right)t^n.$$

Now consider the third and final term on the right hand side of (4.3). Similar to the above, we have
(4.9)

$$\sum_{n=k_2}^\infty\left(\sum_{j=0}^{n-k_2}M^{(j)}_{k_1,k_2;II}\right)t^n=\sum_{n=k_2+k_1}^{2k_2-1}\left(\sum_{j=k_1}^{n-k_2}M^{(j)}_{k_1}\right)t^n+\sum_{n=2k_2}^\infty\left(\sum_{j=k_1}^{k_2-1}M^{(j)}_{k_1}\right)t^n+$$
$$\sum_{n=2k_2}^\infty\left(\sum_{j=k_2}^{n-k_2}M^{(j)}_{k_1,k_2;II}\right)t^n,$$

and note that the second term on the right hand side of (4.9) can be written as

$$\sum_{n=2k_2}^\infty\left(\sum_{j=k_1}^{k_2-1}M^{(j)}_{k_1}\right)t^n=\left(\sum_{j=k_1}^{k_2-1}M^{(j)}_{k_1}\right)\frac{t^{2k_2}}{1-t},\tag{4.10}$$

and the final term on the right hand side of (4.9) can be written as
(4.11)

$$t^{k_2}\,\frac{m_{k_1,k_2}(t)}{1-t}=t^{k_2}\left(\sum_{r=0}^\infty t^r\right)\left(\sum_{j=k_2}^\infty M^{(j)}_{k_1,k_2;II}t^j\right)=\sum_{n=2k_2}^\infty\left(\sum_{j=k_2}^{n-k_2}M^{(j)}_{k_1,k_2;II}\right)t^n.$$

From (4.3)-(4.11), we conclude that

$$2tm'_{k_1,k_2}(t)-(k_1+k_2-2)m_{k_1,k_2}(t)=\frac{\big[(k_2-k_1+1)-(k_2-k_1)t\big]k_1t^{k_2}}{(1-t)^2}+\frac{k_2t^{k_2}}{(1-t)^2}+$$
$$2\sum_{n=\max(k_2,2k_1)}^{k_2+k_1-1}\left(\sum_{j=k_1}^{n-k_1}M^{(j)}_{k_1}\right)t^n+2\left(\sum_{j=k_1}^{k_2-1}M^{(j)}_{k_1}\right)\frac{t^{k_2+k_1}}{1-t}+2t^{k_1}\,\frac{m_{k_1,k_2}(t)}{1-t}+$$
$$2\sum_{n=k_2+k_1}^{2k_2-1}\left(\sum_{j=k_1}^{n-k_2}M^{(j)}_{k_1}\right)t^n+2\left(\sum_{j=k_1}^{k_2-1}M^{(j)}_{k_1}\right)\frac{t^{2k_2}}{1-t}+2t^{k_2}\,\frac{m_{k_1,k_2}(t)}{1-t},$$

which we write as

(4.12)
$$\begin{aligned}
&m'_{k_1,k_2}(t) = A(t)m_{k_1,k_2}(t) + B_1(t) + B_2(t), \text{ where}\\
&A(t) = \frac{k_1+k_2-2}{2t} + \frac{t^{k_1-1}+t^{k_2-1}}{1-t};\\
&B_1(t) = \big[k_2 + k_1(k_2-k_1+1) - k_1(k_2-k_1)t\big]\frac{t^{k_2-1}}{2(1-t)^2};\\
&B_2(t) = \sum_{n=\max(k_2,2k_1)}^{k_2+k_1-1}\Big(\sum_{j=k_1}^{n-k_1} M^{(j)}_{k_1}\Big)t^{n-1} + \sum_{n=k_2+k_1}^{2k_2-1}\Big(\sum_{j=k_1}^{n-k_2} M^{(j)}_{k_1}\Big)t^{n-1}+\\
&\left(\sum_{j=k_1}^{k_2-1} M^{(j)}_{k_1}\right)\frac{t^{k_2+k_1-1}+t^{2k_2-1}}{1-t},
\end{aligned}$$

the dependence of A, B_1 and B_2 on k_1 and k_2 being suppressed. Solving the linear ODE gives for any $\epsilon \in (0,1)$,

(4.13)
$$m_{k_1,k_2}(t) = m_{k_1,k_2}(\epsilon)e^{\int_\epsilon^t A(s)ds} + \int_\epsilon^t e^{\int_s^t A(r)dr}\left(B_1(s)+B_2(s)\right)ds,\ \epsilon < t < 1.$$

Writing $\frac{t^l}{1-t} = \frac{1}{1-t} - \sum_{j=0}^{l-1} t^j$, we have $\int \frac{t^l}{1-t}dt = -\log(1-t) - \sum_{j=1}^{l}\frac{t^j}{j}$. Using this with the formula for A in (4.12), we obtain

$$\int A(r)dr = \frac{k_1+k_2-2}{2}\log t - 2\log(1-t) - \sum_{j=1}^{k_1-1}\frac{t^j}{j} - \sum_{j=1}^{k_2-1}\frac{t^j}{j},$$

and thus

(4.14)
$$e^{\int_s^t A(r)dr} = \left(\frac{t}{s}\right)^{\frac{1}{2}(k_1+k_2-2)}\frac{(1-s)^2}{(1-t)^2}\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-t^j}{j} + \sum_{j=1}^{k_2-1}\frac{s^j-t^j}{j}\right).$$

By (4.4), $m_{k_1,k_2}(\epsilon) = O(\epsilon^{k_2})$, thus it follows from (4.14) that $\lim_{\epsilon\to 0} m_{k_1,k_2}(\epsilon)e^{\int_\epsilon^t A(s)ds} = 0$. Thus, from (4.12)-(4.14) we conclude that

(4.15)
$$\begin{aligned}
&m_{k_1,k_2}(t) = \frac{t^{\frac{1}{2}(k_1+k_2-2)}}{(1-t)^2}\times\\
&\int_0^t \frac{(1-s)^2}{s^{\frac{1}{2}(k_1+k_2-2)}}\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-t^j}{j} + \sum_{j=1}^{k_2-1}\frac{s^j-t^j}{j}\right)\left(B_1(s)+B_2(s)\right)ds.
\end{aligned}$$

Similar to the proof of Theorem 2, we conclude from (4.15) that
(4.16)
$$\lim_{n\to\infty}\frac{M^{(n)}_{k_1,k_2;II}}{n}=\lim_{t\to1^-}(1-t)^2m_{k_1,k_2}(t)=$$
$$\int_0^1\frac{(1-s)^2}{s^{\frac12(k_1+k_2-2)}}\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)(B_1(s)+B_2(s))\,ds,$$

where B_1,B_s are as in (4.12). Substituting for B_1 from (4.12), we have after some algebra,
(4.17)
$$\int_0^1\frac{(1-s)^2}{s^{\frac12(k_1+k_2-2)}}\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)B_1(s)ds=$$
$$\frac12\int_0^1 s^{\frac{k_2-k_1}{2}}\big(k_1+k_2+k_1(k_2-k_1)(1-s)\big)\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)ds.$$

One can easily verify that $\frac{(1-s)^2}{s^{\frac12(k_1+k_2-2)}}B_2(s)$, where B_2 is as in (4.12), is equal to $B(s)$, where B is as in (1.14). Using this with (4.16) and (4.17) proves Theorem 5. □

5. Proof of Theorem 6

We write the second term on the right hand side of (1.11) as

$$\sum_{l=k_1}^{k_2-1}g_{k_2;l}M^{(l)}_{k_1}=m_{k_2}\frac{k_2-1}{m_{k_2}}\sum_{l:\frac{l}{k_2-1}\in[\frac{k_1}{k_2-1},1]}\frac{l}{k_2-1}g_{k_2;l}\frac{M^{(l)}_{k_1}}{l}.\tag{5.1}$$

By (1.5), Theorem 3 and Corollary 1,

$$\lim_{k_2\to\infty}\frac{k_2-1}{m_{k_2}}\sum_{l:\frac{l}{k_2-1}\in[\frac{k_1}{k_2-1},1]}g_{k_2;l}\frac{l}{k_2-1}=\int_0^1\gamma f_{\mu_\infty^{\text{gaps}}}(\gamma)d\gamma=\frac{1-m_\infty}{m_\infty}.\tag{5.2}$$

By (1.1), $\lim_{l\to\infty}\frac{M^{(l)}_{k_1}}{l}=m_{k_1}$, and of course $\lim_{k_2\to\infty}m_{k_2}=m_\infty$. Using this with (5.1), (5.2) and (1.11) proves Theorem 6. □

6. Proof of Theorem 7

We break up the formula for $m_{k_1,k_2;II}$ in (1.13) into four parts. The first part is $m_{k_1,k_2;II_1}$ from (1.12):

$$I_{k_2} := m_{k_1,k_2;II_1} =$$
$$\frac{1}{2}\int_0^1 \exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)s^{\frac{k_2-k_1}{2}}\big(k_1+k_2+k_1(k_2-k_1)(1-s)\big)ds.$$

Recalling (1.14), we define the three other parts by

$$II_{k_2}=\big(\sum_{j=k_1}^{k_2-1}M_{k_1}^{(j)}\big)\int_0^1 \exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)(1-s)\left(s^{\frac{1}{2}k_2+\frac{1}{2}k_1}+s^{\frac{3}{2}k_2-\frac{1}{2}k_1}\right)ds,$$

$$III_{k_2}=\int_0^1 \exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)(1-s)^2\sum_{n=k_2}^{k_2+k_1-1}\big(\sum_{j=k_1}^{n-k_1}M_{k_1}^{(j)}\big)s^{n-\frac{k_1}{2}-\frac{k_2}{2}}ds,$$

and

$$IV_{k_2}=\int_0^1 \exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)(1-s)^2\sum_{n=k_2+k_1}^{2k_2-1}\big(\sum_{j=k_1}^{n-k_2}M_{k_1}^{(j)}\big)s^{n-\frac{k_1}{2}-\frac{k_2}{2}}ds.$$

(The lower bound in the outer summation in III_{k_2} is actually $\max(k_2,2k_1)$, but since we are analyzing the case that k_1 is fixed and $k_2\to\infty$, it suffices to write k_2.)

We first consider I_{k_2}. Making the same substitutions as were made in (1.21) and (1.22), with k_2 in place of k so that $x=k_2(1-s)$, we have

$$\begin{aligned}
&k_2\int_a^1 \exp\big(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\big)s^{\frac{k_2}{2}}ds=\\
(6.1)\qquad &\int_0^{(1-a)k_2}\exp\left(-\int_0^x\frac{1-(1-\frac{y}{k_2})^{k_2-1}}{y}dy\right)(1-\frac{x}{k_2})^{\frac{k_2}{2}}dx\overset{k_2\to\infty}{\to}\\
&\int_0^\infty \exp\left(-\int_0^x\frac{1-e^{-y}}{y}dy\right)e^{-\frac{x}{2}}dx,\ \text{for any } a\in[0,1),
\end{aligned}$$

where the limit follows from the dominated convergence theorem. From (6.1) it follows that

$$\lim_{k_2\to\infty} k_2\int_0^1 h(s)\exp\big(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\big)s^{\frac{k_2}{2}}ds=h(1)\int_0^\infty \exp\left(-\int_0^x\frac{1-e^{-y}}{y}dy\right)e^{-\frac{x}{2}}\,dx, \tag{6.2}$$

for any continuous function h on $[0,1]$.

From (6.2) and the definition of I_{k_2} it follows that

$$\begin{aligned}&\lim_{k_2\to\infty} I_{k_2}=\frac{1}{2}\lim_{k_2\to\infty} k_2\int_0^1\exp\big(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\big)s^{\frac{k_2}{2}}ds=\\ &\frac{1}{2}\int_0^\infty \exp\left(-\int_0^x\frac{1-e^{-y}}{y}dy\right)e^{-\frac{x}{2}}\,dx=D,\end{aligned} \tag{6.3}$$

where D is as in (1.18). This proves (1.16) and (1.18).

To prove (1.17) and thereby complete the proof of the theorem, we need to show that

$$\lim_{k_2\to\infty}(II_{k_2}+III_{k_2}+IV_{k_2})=(1-D)m_{k_1}. \tag{6.4}$$

We first consider II_{k_2}. Similar to (6.1), we have

$$\begin{aligned}&k_2^2\int_a^1\exp\big(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\big)(1-s)s^{bk_2}ds=\\ &\int_0^{(1-a)k_2}\exp\left(-\int_0^x\frac{1-(1-\frac{y}{k_2})^{k_2-1}}{y}dy\right)x(1-\frac{x}{k_2})^{bk_2}dx\stackrel{k_2\to\infty}{\to}\\ &\int_0^\infty\exp\left(-\int_0^x\frac{1-e^{-y}}{y}dy\right)xe^{-bx}dx,\ \text{for } b>0 \text{ and any } a\in[0,1),\end{aligned} \tag{6.5}$$

from which it follows that

$$\begin{aligned}&\lim_{k_2\to\infty}k_2^2\int_0^1 h(s)\exp\big(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\big)(1-s)s^{bk_2}ds=\\ &h(1)\int_0^\infty\exp\left(-\int_0^x\frac{1-e^{-y}}{y}dy\right)xe^{-bx}dx,\\ &\text{for } b>0 \text{ and any continuous function } h \text{ on } [0,1].\end{aligned} \tag{6.6}$$

By (1.1), $\sum_{j=k_1}^{k_2-1} M_{k_1}^{(j)} \sim \sum_{j=k_1}^{k_2-1} jm_{k_1} \sim m_{k_1}\frac{k_2^2}{2}$, as $k_2 \to \infty$. Using this with (6.6) and the definition of II_{k_2}, it follows that

(6.7)
$$\lim_{k_2\to\infty} II_{k_2} = \frac{m_{k_1}}{2}\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right) x\left(e^{-\frac{1}{2}x}+e^{-\frac{3}{2}x}\right)dx.$$

We now consider III_{k_2}. By (1.1), $\sum_{j=k_1}^{n-k_1} M_{k_1}^{(j)} \sim \sum_{j=k_1}^{n-k_1} jm_{k_1} \sim m_{k_1}\frac{k_2^2}{2}$ as $k_2 \to \infty$, uniformly over $n \in \{k_2, \cdots, k_2+k_1-1\}$. Thus, we have

$$\sum_{n=k_2}^{k_2+k_1-1}\Big(\sum_{j=k_1}^{n-k_1} M_{k_1}^{(j)}\Big)s^{n-\frac{k_1}{2}-\frac{k_2}{2}} \sim k_1 m_{k_1}\frac{k_2^2}{2}s^{\frac{k_2}{2}}h_{k_1}(s), \text{ as } k_2\to\infty,$$

where $h_{k_1}(s)$ is continuous on $[0,1]$.

Thus,

(6.8)
$$III_{k_2} \sim k_1 m_{k_1}\frac{k_2^2}{2}\int_0^1 \exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)(1-s)^2 s^{\frac{k_2}{2}}h_{k_1}(s)ds.$$

Making the same substitutions as before, we have

$$\begin{aligned} &k_2^2\int_0^1 \exp\left(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)(1-s)^2 s^{\frac{k_2}{2}}ds = \\ &\frac{1}{k_2}\int_0^{k_2}\exp\left(-\int_0^x \frac{1-(1-\frac{y}{k_2})^{k_2-1}}{y}dy\right)x^2(1-\frac{x}{k_2})^{\frac{k_2}{2}}dx \overset{k_2\to\infty}{\to} 0. \end{aligned} \tag{6.9}$$

From (6.8) and (6.9), we conclude that

$$\lim_{k_2\to\infty} III_{k_2} = 0. \tag{6.10}$$

We now consider IV_{k_2}, which is somewhat more involved. From the substitutions in (1.21) and (1.22) that were applied to obtain (6.1) and (6.5), and from the conclusions in (6.2) and (6.6), it is easy to see that the following holds.

For $h(x)$ continuous on $[0,1]$ with $h(1)>0$ and $b>0$,

$$\lim_{k_2\to\infty} k_2^C \int_0^1 \exp\big(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\big)(1-s)^D s^{bk_2}h(s)ds = \tag{6.11}$$
$$\begin{cases} h(1)\int_0^\infty \exp\Big(-\int_0^x \frac{1-e^{-y}}{y}dy\Big)\,x^D e^{-bx}dx, & \text{if } C=D+1;\\ \infty, & \text{if } C>D+1;\\ 0, & \text{if } C<D+1.\end{cases}$$

Because of this, when calculating $\lim_{k_2\to\infty} IV_{k_2}$, we may replace the term $M^{(j)}_{k_1}$ by the expression for its leading asymptotic behavior as $j\to\infty$, namely jm_{k_1}, as follows from (1.1). We can also let $h(s)=s^{-\frac{k_1}{2}}\exp(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j})$, which satisfies $h(1)=1$. Thus, we have

$$\lim_{k_2\to\infty} IV_{k_2} = \tag{6.12}$$
$$\lim_{k_2\to\infty}\int_0^1 \exp(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j})(1-s)^2 \sum_{n=k_2+k_1}^{2k_2-1}\big(\sum_{j=k_1}^{n-k_2} jm_{k_1}\big)s^{n-\frac{k_2}{2}}ds.$$

The same considerations in fact allow us to conclude that

$$\lim_{k_2\to\infty} IV_{k_2} = \tag{6.13}$$
$$\lim_{k_2\to\infty}\int_0^1 \exp(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j})(1-s)^2 \sum_{n=k_2}^{2k_2-1}\big(\sum_{j=0}^{n-k_2} jm_{k_1}\big)s^{n-\frac{k_2}{2}}ds,$$

because the difference between $\sum_{n=k_2}^{2k_2-1}\big(\sum_{j=0}^{n-k_2} jm_{k_1}\big)s^{n-\frac{k_2}{2}}$ and $\sum_{n=k_2+k_1}^{2k_2-1}\big(\sum_{j=k_1}^{n-k_2} jm_{k_1}\big)s^{n-\frac{k_2}{2}}$ is of lower order than $\sum_{n=k_2+k_1}^{2k_2-1}\big(\sum_{j=k_1}^{n-k_2} jm_{k_1}\big)s^{n-\frac{k_2}{2}}$ as $k_2\to\infty$. We prefer to work with (6.13) rather than with (6.12) because this makes the calculations below somewhat simpler.

We write

$$\sum_{n=k_2}^{2k_2-1}\big(\sum_{j=0}^{n-k_2} jm_{k_1}\big)s^{n-\frac{k_2}{2}} = \frac{m_{k_1}}{2}\sum_{n=k_2}^{2k_2-1}(n-k_2)(n-k_2+1)s^{n-\frac{k_2}{2}}, \tag{6.14}$$

and

(6.15)

$$\sum_{n=k_2}^{2k_2-1}(n-k_2)(n-k_2+1)s^{n-\frac{k_2}{2}} = s^{\frac{k_2}{2}}\sum_{m=0}^{k_2-1} m(m+1)s^m =$$
$$s^{\frac{k_2}{2}}\sum_{m=-1}^{k_2-1} m(m+1)s^m = s^{\frac{k_2}{2}+1}(\sum_{m=-1}^{k_2-1} s^{m+1})'' = s^{\frac{k_2}{2}+1}(\frac{1-s^{k_2+1}}{1-s})'' =$$
$$s^{\frac{k_2}{2}+1}\left(\frac{2(1-s^{k_2+1})}{(1-s)^3} - \frac{2(k_2+1)s^{k_2}}{(1-s)^2} - \frac{k_2(k_2+1)s^{k_2-1}}{1-s}\right).$$

From (6.13)-(6.15) we have

(6.16)

$$\lim_{k_2\to\infty} IV_{k_2} = \frac{m_{k_1}}{2}\lim_{k_2\to\infty}\int_0^1 \exp(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j})\times$$
$$\left(\frac{2(s^{\frac{k_2}{2}+1}-s^{\frac{3k_2}{2}+2})}{(1-s)} - 2(k_2+1)s^{\frac{3k_2}{2}+1} - k_2(k_2+1)(1-s)s^{\frac{3k_2}{2}}\right)ds.$$

Making the same substitutions as before, we obtain

(6.17)

$$\lim_{k_2\to\infty}\int_0^1 \exp(\sum_{j=1}^{k_2-1}\frac{s^j-1}{j})\times$$
$$\left(\frac{2(s^{\frac{k_2}{2}+1}-s^{\frac{3k_2}{2}+2})}{(1-s)} - 2(k_2+1)s^{\frac{3k_2}{2}+1} - k_2(k_2+1)(1-s)s^{\frac{3k_2}{2}}\right)ds =$$
$$\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)\left(\frac{2(e^{-\frac{x}{2}}-e^{-\frac{3x}{2}})}{x} - 2e^{-\frac{3x}{2}} - xe^{-\frac{3x}{2}}\right)dx.$$

From (6.16) and (6.17) we conclude that

(6.18)

$$\lim_{k_2\to\infty} IV_{k_2} = \frac{m_{k_1}}{2}\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)\left(\frac{2(e^{-\frac{x}{2}}-e^{-\frac{3x}{2}})}{x} - 2e^{-\frac{3x}{2}} - xe^{-\frac{3x}{2}}\right)dx.$$

Note that

$$\left(\exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)xe^{-\frac{1}{2}x}\right)' = -\exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)\left(\frac{1}{2}xe^{-\frac{1}{2}x} - e^{-\frac{3}{2}x}\right).$$

Thus,

$$\begin{aligned}(6.19)\qquad &\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)\left(\frac{1}{2}xe^{-\frac{1}{2}x}-e^{-\frac{3}{2}x}\right)dx=\\ &-\left(\exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)xe^{-\frac{1}{2}x}\right)\Big|_0^\infty=0.\end{aligned}$$

Also note that the term $\frac{m_{k_1}}{2}\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)xe^{-\frac{3}{2}x}dx$ in (6.7) cancels with a term from (6.18). Using this last fact along with (6.7), (6.10), (6.18) and (6.19), we obtain

(6.20)

$$\lim_{k_2\to\infty}(II_{k_2}+III_{k_2}+IV_{k_2})=m_{k_1}\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)\frac{(e^{-\frac{x}{2}}-e^{-\frac{3x}{2}})}{x}dx.$$

Integrating by parts with $u=e^{-\frac{1}{2}x}$ and $v'=\exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)\frac{1-e^{-x}}{x}$, we have

(6.21)

$$\begin{aligned}&\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)\frac{(e^{-\frac{x}{2}}-e^{-\frac{3x}{2}})}{x}dx=\\ &-e^{-\frac{1}{2}x}\exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)\Big|_0^\infty-\frac{1}{2}\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)e^{-\frac{1}{2}x}dx=\\ &1-\frac{1}{2}\int_0^\infty \exp\left(-\int_0^x \frac{1-e^{-y}}{y}dy\right)e^{-\frac{1}{2}x}dx.\end{aligned}$$

Now (6.4) follows from (6.20) and (6.21).

□

7. Proof of Theorem 8

Recalling that $M_{k_1}^{(l)}=k_1$, for $l\in[k_1,2k_1-1]$, we have from (1.11),

(7.1)

$$m_{k_1,[Lk_1];I}=m_{[Lk_1]}+\sum_{l=k_1}^{[Lk_1]-1}g_{[Lk_1];l}M_{k_1}^{(l)}=m_{[Lk_1]}+k_1\sum_{l=k_1}^{[Lk_1]-1}g_{[Lk_1];l},\ L\in(1,2].$$

Substituting from (1.3) in the sum on the right hand side of (7.1), we have

$$\begin{aligned}(7.2)\qquad &k_1\sum_{l=k_1}^{[Lk_1]-1}g_{[Lk_1];l}=2k_1\int_0^1(1-s)\left(\sum_{l=k_1}^{[Lk_1]-1}s^l\right)e^{2\sum_{j=1}^{[Lk_1]-1}\frac{s^j-1}{j}}ds=\\ &2k_1\int_0^1\left(s^{k_1}-s^{[Lk_1]}\right)e^{2\sum_{j=1}^{[Lk_1]-1}\frac{s^j-1}{j}}ds.\end{aligned}$$

Making the same substitutions as were made in (1.21) and (1.22), with $[Lk_1]$ in place of k so that $x = [Lk_1](1-s)$, we obtain from (7.2)

(7.3)
$$k_1 \sum_{l=k_1}^{[Lk_1]-1} g_{[Lk_1];l} =$$
$$\frac{2}{L}\int_0^{[Lk_1]} \left((1-\frac{x}{[Lk_1]})^{k_1} - (1-\frac{x}{[Lk_1]})^{[Lk_1]}\right) \exp\left(-2\int_0^x \frac{1-(1-\frac{y}{[Lk_1]})^{[Lk_1]-1}}{y} dy\right) dx \overset{k_1\to\infty}{\to}$$
$$\frac{2}{L}\int_0^{\infty} \left(e^{-\frac{1}{L}x} - e^{-x}\right) e^{-2\int_0^x \frac{1-e^{-y}}{y}dy} dx,$$

which proves (1.19). □

8. Proof of Theorem 9

We begin by writing down explicitly $B(s)$ from (1.14) in the case that $k_2 = [Lk_1]$ with $L \in (1,2]$. Since $L \in (1,2]$, we have $M_{k_1}^{(j)} = k_1$ in every place it appears on the right hand side of (1.14). We continue to write k_2 instead of $[Lk_1]$, in order to have the long calculations below take up a little less space. We have

$$\begin{aligned} &B(s) = (1-s)^2 s^{-\frac{k_1}{2}-\frac{k_2}{2}} \times \\ &\left(k_1 \sum_{n=2k_1}^{k_2+k_1-1} (n-2k_1+1)s^n + k_1 \sum_{n=k_2+k_1}^{2k_2-1} \left(n-k_2-k_1+1\right)s^n\right) + \\ &k_1\left(k_2-k_1\right)(1-s)\left(s^{\frac{3k_2}{2}-\frac{k_1}{2}} + s^{\frac{k_2}{2}+\frac{k_1}{2}}\right). \end{aligned} \tag{8.1}$$

Differentiating and performing some algebra, we have

$$\begin{aligned} &\sum_{n=a}^{b} ns^n = s\left(\sum_{n=a}^{b} s^n\right)' = \\ &\frac{s}{(1-s)^2}\left(as^{a-1} - (a-1)s^a - (b+1)s^b + bs^{b+1}\right), \ a<b. \end{aligned} \tag{8.2}$$

Using (8.2), the two sums on the right hand side of (8.1) can be written as

(8.3)

$$\sum_{n=2k_1}^{k_2+k_1-1}(n-2k_1+1)s^n=\frac{s}{(1-s)^2}\times$$
$$\Big(2k_1s^{2k_1-1}-(2k_1-1)s^{2k_1}-(k_2+k_1)s^{k_2+k_1-1}+(k_2+k_1-1)s^{k_2+k_1}\Big)-$$
$$\frac{2k_1-1}{1-s}\left(s^{2k_1}-s^{k_2+k_1}\right);$$
$$\sum_{n=k_2+k_1}^{2k_2-1}(n-k_2-k_1+1)s^n=\frac{s}{(1-s)^2}\times$$
$$\Big((k_2+k_1)s^{k_2+k_1-1}-(k_2+k_1-1)s^{k_2+k_1}-2k_2s^{2k_2-1}+(2k_2-1)s^{2k_2}\Big)-$$
$$\frac{k_2+k_1-1}{1-s}\left(s^{k_2+k_1}-s^{2k_2}\right).$$

Substituting (8.3) in (8.1), we obtain

$$B(s)=$$
$$k_1\left(2k_1s^{\frac32k_1-\frac12k_2}-(2k_1-1)s^{\frac32k_1-\frac{k_2}{2}+1}-(k_2+k_1)s^{\frac{k_2}{2}+\frac{k_1}{2}}+(k_2+k_1-1)s^{\frac{k_2}{2}+\frac{k_1}{2}+1}\right)-$$
$$k_1(2k_1-1)(1-s)\left(s^{\frac32k_1-\frac{k_2}{2}}-s^{\frac{k_2}{2}+\frac{k_1}{2}}\right)+$$
$$k_1\left((k_2+k_1)s^{\frac{k_2}{2}+\frac{k_1}{2}}-(k_2+k_1-1)s^{\frac{k_2}{2}+\frac{k_1}{2}+1}-2k_2s^{\frac32k_2-\frac12k_1}+(2k_2-1)s^{\frac32k_2-\frac12k_1+1}\right)-$$
$$k_1(k_2+k_1-1)(1-s)\left(s^{\frac{k_2}{2}+\frac{k_1}{2}}-s^{\frac32k_2-\frac12k_1}\right)+$$
$$k_1\left(k_2-k_1\right)(1-s)\left(s^{\frac{3k_2}{2}-\frac{k_1}{2}}+s^{\frac{k_2}{2}+\frac{k_1}{2}}\right).$$

Collecting terms, one finds that the above calculation reduces to

$$B(s)=k_1(s^{\frac32k_1-\frac12k_2}-s^{\frac32k_2-\frac12k_1}). \tag{8.4}$$

From (1.13), (1.12) and (8.4), we conclude that

(8.5)

$$m_{k_1,k_2;II}=$$
$$\frac12\int_0^1 s^{\frac{k_2-k_1}{2}}\big(k_1+k_2+k_1(k_2-k_1)(1-s)\big)\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)ds+$$
$$\int_0^1 k_1(s^{\frac32k_1-\frac12k_2}-s^{\frac32k_2-\frac12k_1})\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)ds,\ \text{if } k_2\le 2k_1.$$

Consider the first term on the right hand side of (8.5) with $k_2 = [Lk_1]$. By the same substitution used in (1.21) and (1.22), with k_1 in place of k so that $x = k_1(1-s)$, we have

(8.6)

$$\lim_{k_1\to\infty}\frac{1}{2}\int_0^1 s^{\frac{k_2-k_1}{2}}\big(k_1+k_2+k_1(k_2-k_1)(1-s)\big)\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)ds =$$

$$\lim_{k_1\to\infty}\frac{1}{2}\int_0^{k_1}(1-\frac{x}{k_1})^{\frac{L-1}{2}k_1}\left(1+L+(L-1)x\right)\times$$

$$\exp\left(-\int_0^x\frac{1-(1-\frac{y}{k})^{k-1}}{y}dy-\int_0^{Lx}\frac{1-(1-\frac{y}{k})^{k-1}}{y}dy\right)dx =$$

$$\frac{1}{2}\int_0^\infty\left(L+1+(L-1)x\right)e^{-\frac{L-1}{2}x}e^{-\int_0^x\frac{1-e^{-y}}{y}dy-\int_0^{Lx}\frac{1-e^{-y}}{y}dy}dx.$$

Similarly, for the second term on the right hand side of (8.5), we have

(8.7)

$$\lim_{k_1\to\infty}\int_0^1 k_1(s^{\frac{3}{2}k_1-\frac{1}{2}k_2}-s^{\frac{3}{2}k_2-\frac{1}{2}k_1})\exp\left(\sum_{j=1}^{k_1-1}\frac{s^j-1}{j}+\sum_{j=1}^{k_2-1}\frac{s^j-1}{j}\right)ds =$$

$$\int_0^\infty\left(e^{-\frac{3-L}{2}x}-e^{-\frac{3L-1}{2}x}\right)e^{-\int_0^x\frac{1-e^{-y}}{y}dy}e^{-\int_0^{Lx}\frac{1-e^{-y}}{y}dy}dx.$$

From (8.5)-(8.7), we obtain (1.20), completing the proof of the theorem. □

Data Availability All data is included in the paper text.

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